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The Maslov index for Lagrangian pairs on $\mathbb{R}^{2n}$

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ABSTRACT

We discuss a definition of the Maslov index for Lagrangian pairs on $\mathbb{R}^{2n}$ based on spectral flow, and develop many of its salient properties. We provide two applications to illustrate how our approach leads to a straightforward analysis of the relationship between the Maslov index and the Morse index for Schrödinger operators on $[0,1]$ and $\mathbb{R}$.

1. Introduction

With origins in the work of V.P. Maslov [16] and subsequent development by V.I. Arnol'd [1], the Maslov index on $\mathbb{R}^{2n}$ is a tool for determining the nature of intersections between two evolving Lagrangian subspaces (see Definition 1.1). As discussed in [6], several equivalent definitions are available, and we focus on a definition for Lagrangian pairs based on the development in [4] (using the definition of spectral flow introduced in [17]). We note at the outset that the theory associated with the Maslov index has now been extended well beyond the simple setting of our analysis (see, for example, [4,7,8,10]); nonetheless, the Maslov index for Lagrangian pairs on $\mathbb{R}^{2n}$ is a useful tool, and a systematic development of its properties is certainly warranted.

As a starting point, we define what we will mean by a *Lagrangian subspace* of $\mathbb{R}^{2n}$.

Definition 1.1. We say $\ell \subset \mathbb{R}^{2n}$ is a Lagrangian subspace if ℓ has dimension n and

$$(Jx, y)_{\mathbb{R}^{2n}} = 0,$$

for all $x, y \in \ell$. Here, $(\cdot, \cdot)_{\mathbb{R}^{2n}}$ denotes Euclidean inner product on $\mathbb{R}^{2n}$, and

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$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix},$$

with I_n the $n \times n$ identity matrix. We sometimes adopt standard notation for symplectic forms, $\omega(x, y) = (Jx, y)_{\mathbb{R}^{2n}}$. Finally, we denote by $\Lambda(n)$ the collection of all Lagrangian subspaces of $\mathbb{R}^{2n}$, and we will refer to this as the *Lagrangian Grassmannian*.

A simple example, important for intuition, is the case $n = 1$, for which $(Jx, y)_{\mathbb{R}^2} = 0$ if and only if x and y are linearly dependent. In this case, we see that any line through the origin is a Lagrangian subspace of $\mathbb{R}^2$. As a foreshadowing of further discussion, we note that each such Lagrangian subspace can be identified with precisely two points on the unit circle S^1.

More generally, any Lagrangian subspace of $\mathbb{R}^{2n}$ can be spanned by a choice of n linearly independent vectors in $\mathbb{R}^{2n}$. We will generally find it convenient to collect these n vectors as the columns of a $2n \times n$ matrix $\mathbf{X}$, which we will refer to as a *frame* for ℓ. Moreover, we will often write $\mathbf{X} = \binom{X}{Y}$, where X and Y are $n \times n$ matrices.

Given any two Lagrangian subspaces ℓ_1 and ℓ_2, with associated frames $\mathbf{X}_1 = \binom{X_1}{Y_1}$ and $\mathbf{X}_2 = \binom{X_2}{Y_2}$, we can define the complex $n \times n$ matrix

$$\tilde{W} = -(X_1 + iY_1)(X_1 - iY_1)^{-1}(X_2 - iY_2)(X_2 + iY_2)^{-1}, \tag{1.1}$$

which we will see in Section 3 is unitary. (We will also verify in Section 3 that $(X_1 - iY_1)$ and $X_2 + iY_2$ are both invertible, and that $\tilde{W}$ is independent of the choice of frames we take for ℓ_1 and ℓ_2.) Notice that if we switch the roles of ℓ_1 and ℓ_2 then $\tilde{W}$ will be replaced by $\tilde{W}^{-1}$, and since $\tilde{W}$ is unitary this is $\tilde{W}^*$. We conclude that the eigenvalues in the switched case will be complex conjugates of those in the original case.

Remark 1.2. We use the tilde to distinguish the $n \times n$ complex-valued matrix $\tilde{W}$ from the Souriau map (see equation (3.8) below), which is a related $2n \times 2n$ matrix often—as here—denoted W. The general form of $\tilde{W}$ appears in a less general context in [9,12]. For the special case $\mathbf{X}_2 = \binom{0}{I}$ (associated, for example, with Dirichlet boundary conditions for a Sturm–Liouville eigenvalue problem) we see that

$$\tilde{W} = (X_1 + iY_1)(X_1 - iY_1)^{-1}, \tag{1.2}$$

which has been extensively studied, perhaps most systematically in [2] (particularly Chapter 10). If we let $\tilde{W}_D$ denote (1.2) for $\mathbf{X}_1 = \binom{0}{I}$ and for $j = 1, 2$ set

$$\tilde{W}_j = (X_j + iY_j)(X_j - iY_j)^{-1},$$

then our form for $\tilde{W}$ can be viewed as the composition map

$$-\tilde{W}_1\tilde{W}_D(\tilde{W}_2\tilde{W}_D)^{-1} = -\tilde{W}_1(\tilde{W}_2)^{-1}. \tag{1.3}$$

For a related observation regarding the Souriau map see Remark 3.3.

Combining observations from Sections 2 and 3, we will establish the following theorem (cf. Lemma 1.3 in [4]).

Theorem 1.3. *Suppose $\ell_1, \ell_2 \subset \mathbb{R}^{2n}$ are Lagrangian subspaces, with respective frames $\mathbf{X}_1 = \binom{X_1}{Y_1}$ and $\mathbf{X}_2 = \binom{X_2}{Y_2}$, and let $\tilde{W}$ be as defined in (1.1). Then*

$$\dim\ker(\tilde{W} + I) = \dim(\ell_1 \cap \ell_2).$$