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Initial behavior of solutions to the Yang–Mills heat equation

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ABSTRACT

We explore the small-time behavior of solutions to the Yang–Mills heat equation with rough initial data. We consider solutions $A(t)$ with initial value $A_0 \in H_{1/2}(M)$, where M is a bounded convex region in $\mathbb{R}^3$ or all of $\mathbb{R}^3$. The behavior, as $t \downarrow 0$, of the $L^p(M)$ norms of the time derivatives of $A(t)$ and its curvature $B(t)$ will be determined for $p = 2$ and 6, along with the $H_1(M)$ norm of these derivatives.

Contents

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1. Introduction

In this article we study the initial behavior of solutions to the Yang–Mills heat equation over a region M in $\mathbb{R}^3$. Denote by K a compact connected Lie group with Lie algebra $\mathfrak{k}$. A $\mathfrak{k}$ valued 1-form over M may be written as

$$A = \sum_{j=1}^{3} A_j(x)dx^j, \tag{1.1}$$

with coefficients $A_j(x) \in \mathfrak{k}$. The curvature of A is the $\mathfrak{k}$ valued 2-form given by $B = dA + A \wedge A$. The Yang–Mills heat equation is the weakly parabolic equation for a time dependent $\mathfrak{k}$ valued 1-form $A(t)$ over M given by

$$\frac{\partial}{\partial t} A(x,t) = -d^*_{A(t)} B(x,t), \tag{1.2}$$

wherein $d^*_A = d^* +$ [the interior product by $ad\, A(t)$], and $B(x,t)$ is the curvature of $A(t)$ at x. We will always take K to be a subgroup of the orthogonal, respectively unitary, group of a finite dimensional real, respectively complex, inner product space $\mathcal{V}$.

The Yang–Mills heat equation is only weakly parabolic since the second order derivative terms on the right side of (1.2) are $-d^*dA$, which are missing 'half' of the Laplacian on 1-forms $-\Delta = d^*d + dd^*$. In [1] we proved the existence and uniqueness of solutions to this equation for initial data A_0 in $H_1(M)$. In [6] the existence and uniqueness was proven for initial data in $H_{1/2}(M)$. The Sobolev index 1/2 is the critical index for the Yang–Mills heat equation in spatial dimension three. We will be concerned with solutions to (1.2) for which the initial value A_0 is in $H_{1/2}(M)$. In this case the curvature $B(t)$ can be expected to blow up in the $L^2(M)$ sense as $t \downarrow 0$ since, informally, $B(t)$ can be expected to converge to its initial value B_0 only in the sense of the negative Sobolev space $H_{-1/2}(M)$. Higher derivatives of $A(t)$ can be expected to blow up more quickly as $t \downarrow 0$. Our study is motivated by a desire to understand the nature of the singularities of gauge covariant derivatives of a solution to the Yang–Mills heat equation as time decreases to zero. In this article we will study the $L^p(M)$ behavior of various gauge covariant derivatives of $A(t)$ as $t \downarrow 0$. The values $p = 2$ and $p = 6$ (and a fortiori all p in between) are of sole interest in this paper because only first order Sobolev inequalities can be effectively used in our energy methods. Concerning higher values of p see Remark 6.14. A priori estimates of first, second and third order spatial covariant derivatives have already been used in our previous work [1,2,6] to prove existence and uniqueness of solutions to (1.2).

A function $g : M \to K$ induces a gauge transformation of a time dependent connection form on M by the definition

$$A^g(x,t) = g(x)^{-1}A(x,t)g(x) + g(x)^{-1}dg(x). \tag{1.3}$$

If $A(\cdot,\cdot)$ is a solution to the Yang–Mills heat equation (1.2) then so is $A^g(\cdot,\cdot)$, at least if g satisfies some mild regularity conditions. It is already clear from this that the Yang–Mills heat equation does not smooth all initial data, for if $A(x,t)$ is a solution with initial value $A_0(x)$ then $A^g(x,t)$ is the solution with initial value $A_0^g(x)$, and consequently, even if $A(x,t)$ is very smooth, the solution $A^g(x,t)$ needs to be no smoother than $g^{-1}dg$. Our goal is to show that solutions to (1.2) are infinitely differentiable in a gauge covariant sense for $t > 0$, even for rough initial data, and to determine the nature of the singularities of the derivatives as $t \downarrow 0$. For the class of initial data that we are interested in, namely $A_0 \in H_{1/2}(M)$, the formula (1.3) suggests that the corresponding class of allowed gauge functions should include functions $g \in H_{3/2}(M)$. A precise definition of this class, which makes it into a complete topological group, will be given in Section 2. With