J. Math. Anal. Appl. 451 (2017) 906–923

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

The least squares estimator of random variables under sublinear expectations

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ARTICLE INFO

Article history:
Received 27 May 2016
Available online 22 February 2017
Submitted by V. Pozdnyakov

Keywords:
Least squares estimator
Conditional expectation
Sublinear expectation
Coherent risk measure
g-Expectation

ABSTRACT

In this paper, we study the least squares estimator for sublinear expectations. Under some mild assumptions, we prove the existence and uniqueness of the least squares estimator. The relationship between the least squares estimator and the conditional coherent risk measure (resp. the conditional g-expectation) is also explored. Then some characterizations of the least squares estimator are given.

1. Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let ξ be a random variable in $L^2_{\mathcal{F}}(P)$ where

$$L^2_{\mathcal{F}}(P) := \{\xi : \Omega \to \mathbb{R};\ \xi \in L^2(P) \text{ and } \xi \text{ is } \mathcal{F}-\text{measurable}\}.$$

If $\mathcal{C} \subset \mathcal{F}$ is a σ-algebra, then the conditional expectation of ξ, denoted by $E_P[\xi|\mathcal{C}]$, is just the solution of the following problem: find a $\hat{\eta} \in L^2_{\mathcal{C}}(P)$ such that

$$E_P[(\xi - \hat{\eta})^2] = \inf_{\eta \in L^2_{\mathcal{C}}(P)} E_P[(\xi - \eta)^2].$$

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1 This work was supported by the Fund of Doctoral Program Research of University of Jinan (No. 160100119).
2 This work was supported by National Natural Science Foundation of China (No. 11571203); Supported by the Programme of Introducing Talents of Discipline to Universities of China (No. B12023).

http://dx.doi.org/10.1016/j.jmaa.2017.02.020
0022-247X/

$\hat{\eta}$ is called the least squares estimator of ξ. The property that the conditional expectation coincides with the least squares estimator is the basis for the filtering theory (see for example Bensoussan [3], Davis [4] or Kallianpur [9]). From another viewpoint, the least squares estimator can also be used as an alternative definition of the conditional expectation.

In recent decades, nonlinear expectations have been proposed and developed rapidly. Various definitions of conditional nonlinear expectations are introduced. For example, Peng studied the g-expectation in [10] and defined the conditional g-expectation $\mathcal{E}_g[\xi|\mathcal{F}_t]$ as the solution of a backward stochastic differential equation at time t. The g-expectation has many good properties including time consistency, i.e., $\forall 0 \leq s \leq t \leq T$, $\mathcal{E}_g[\mathcal{E}_g[\xi|\mathcal{F}_t]|\mathcal{F}_s] = \mathcal{E}_g[\xi|\mathcal{F}_s]$. Artzner et al. [1] studied the coherent risk measures which can be seen as one kind of nonlinear expectations (see [6] and [12]). The conditional coherent risk measure in [2] is defined as

$$\bar{\Phi}_t[\xi] := \operatorname*{ess\,sup}_{P \in \mathcal{P}} E_P[\xi|\mathcal{F}_t],$$

where $\xi \in \mathcal{F}_T$, $\mathcal{F}_t$ is a sub-σ-algebra of $\mathcal{F}_T$ and $\mathcal{P}$ is a family of probability measures. They have proved that if $\mathcal{P}$ is 'm-stable', then the conditional coherent risk measure defined above is time consistent.

For the nonlinear expectation case, we do not know whether conditional nonlinear expectations still coincide with the least squares estimators. So it is interesting to explore the relationship between the least squares estimators and the nonlinear conditional expectations. In this paper, we focus on the sublinear expectation case. The least squares estimator under sublinear expectations is investigated.

For a sublinear expectation ρ, since ρ can be represented as a supremum of a family of linear expectations, then our least squares estimate problem can be seen as a minimax problem. The key tool to solve these problems is the minimax theorem. We assume ρ is continuous from above in order that the minimax theorem holds. Then the existence and uniqueness of the least squares estimator are obtained. If we denote $\rho(\xi|\mathcal{C})$ as the least squares estimator, note that for any real number λ, $\rho(\lambda\xi|\mathcal{C})) = \lambda\rho(\xi|\mathcal{C})$. This result induces, generally speaking, both of Artzner et al.'s and Peng's nonlinear conditional expectations fail to be least squares estimators.

The problem comes naturally: when least squares estimators will coincide with Artzner et al.'s and Peng's conditional expectations. To give the answer, we investigate the relationship between the least squares estimators and the nonlinear conditional expectations. A sufficient and necessary condition for the coincidence is given.

In the rest part, through variational methods, we show to get the least squares estimator is equivalent to obtain the solution of an associated nonlinear equation.

The paper is organized as follows. In section 2, we formulate our problem. Under some mild assumptions, we prove the existence and uniqueness of the least squares estimator in section 3. In section 4, we first give the basic properties of the least squares estimator. Then we explore the relationship between the least squares estimator and the conditional coherent risk measure and conditional g-expectation. In the last section, we obtain several characterizations of the least squares estimator.

2. Problem formulation

2.1. Preliminary

For a given measurable space $(\Omega, \mathcal{F})$, we denote $\mathbb{F}$ as the set of bounded $\mathcal{F}$-measurable functions, $\mathbb{N}$ as the set of natural numbers and $\mathbb{R}$ as the set of real numbers.

Definition 2.1. A sublinear expectation ρ is a functional $\rho : \mathbb{F} \mapsto \mathbb{R}$ satisfying

(i) Monotonicity: $\rho(\xi_1) \geq \rho(\xi_2)$ if $\xi_1 \geq \xi_2$;
(ii) Constant preserving: $\rho(c) = c$ for $c \in \mathbb{R}$;