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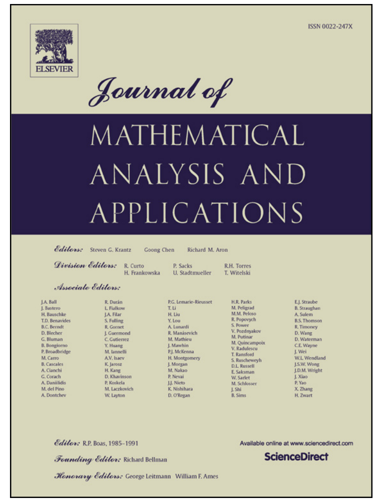

Some evaluation of parametric Euler sums

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Abstract In this paper, by using the method of Contour Integral Representations and the Theorem of Residues and integral representations of series, we discuss the analytic representations of parametric Euler sums that involve harmonic numbers through zeta values and rational function series, either linearly or nonlinearly. Furthermore, we give explicit formulae for several parametric quadratic and cubic sums in terms of zeta values and rational series. Moreover, some interesting new consequences and illustrative examples are considered.

Keywords Harmonic number; Euler sum; Riemann zeta function; Hurwitz zeta function.

AMS Subject Classifications (2010): 11M06; 11M32.

1 Introduction

Throughout this article we will use the following definitions and notations. Let $\mathbb{N} := \{1, 2, 3, \ldots\}$ be the set of natural numbers, and $\mathbb{N}_0 := \mathbb{N} \bigcup \{0\}, \mathbb{N}\backslash\{1\} := \{2, 3, 4, \cdots\}$. In this paper, harmonic numbers, alternating harmonic numbers and their generalizations are classically defined by

$$\zeta_n(k) := \sum_{j=1}^{n} \frac{1}{j^k},\ L_n(k) := \sum_{j=1}^{n} \frac{(-1)^{j-1}}{j^k}, k \in \mathbb{N}, \tag{1.1}$$

where $H_n := \zeta_n(1) = \sum\limits_{j=1}^{n} \frac{1}{j}$ is classical harmonic number and the empty sum $\zeta_0(m)$ is conventionally understood to be zero. The subject of this paper is Euler sums, which are the infinite sums whose general term is a product of harmonic numbers and alternating harmonic numbers of index n and a power of n^{-1} or $((-1)^{n-1}n^{-1})$. Hence, more generally we can define the Euler sums by the series ([18,19])

$$\sum_{n=1}^{\infty} \frac{\prod\limits_{i=1}^{m_1} \zeta_n^{q_i}(k_i) \prod\limits_{j=1}^{m_2} L_n^{l_j}(h_j)}{n^p},\ \sum_{n=1}^{\infty} \frac{\prod\limits_{i=1}^{m_1} \zeta_n^{q_i}(k_i) \prod\limits_{j=1}^{m_2} L_n^{l_j}(h_j)}{n^p}(-1)^{n-1}, \tag{1.2}$$

where $m_1, m_2, q_i, k_i, h_j, l_j, p (p \geq 2)$ are positive integers. The quantity $w := \sum\limits_{i=1}^{m_1}(k_i q_i) + \sum\limits_{j=1}^{m_2}(h_j l_j) + p$ is called the weight, the quantity $k := \sum\limits_{i=1}^{m_1} q_i + \sum\limits_{j=1}^{m_2} l_j$ is called the degree.

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