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Sharp bounds for composition with quasiconformal mappings in Sobolev spaces

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Let ϕ be a quasiconformal mapping, and let T_ϕ be the composition operator which maps f to $f \circ \phi$. Since ϕ may not be bi-Lipschitz, the composition operator need not map Sobolev spaces to themselves. The study begins with the behavior of T_ϕ on L^p and $W^{1,p}$ for $1 < p < \infty$. This cases are well understood but alternative proofs of some known results are provided. Using interpolation techniques it is seen that compactly supported Bessel potential functions in $H^{s,p}$ are sent to $H^{s,q}$ whenever $0 < s < 1$ for appropriate values of q. The techniques used lead to sharp results and they can be applied to Besov spaces as well.

1. Introduction

Given a quasiconformal homeomorphism $\phi : \Omega \to \Omega'$ between domains in $\mathbb{R}^n$, we consider the composition operator T_ϕ which maps every measurable function $f : \Omega' \to \mathbb{R}$ to $f \circ \phi$. It is well known that ϕ lies in a certain Sobolev space $W^{1,p}_{loc}$ with $p > n$, that is, the space of locally p-integrable functions with locally p-integrable derivatives, and in some Hölder class C^s with $0 < s < 1$, i.e. for any $K \subset \Omega$ compact ϕ is bounded and continuous in K with $|\phi(x) - \phi(y)| \le C_K|x - y|^s$ for $x, y \in K$. The composition operator T_ϕ is a self-map of $W^{1,n}$. However, since ϕ may not be bi-Lipschitz, the composition operator does not necessarily map other Sobolev spaces to themselves.

A characterization of homeomorphisms which give rise to bounded composition operators is given in [17] and [12].

Over the last decade, the study on the stability of the planar Calderón inverse conductivity problem raised questions on the range of T_ϕ. The ground-breaking work of Astala and Päivärinta [3] showing uniqueness of the solution to the problem was adapted by Barceló, Faraco and Ruiz in [4] to provide stability in Lipschitz domains with only Hölder a priori conditions on the conductivities. Some years later, Clop, Faraco and

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Ruiz weakened the a priori assumption on the conductivities to just a fractional Sobolev condition in [6], allowing the method to be applied to non-continuous conductivities, and later on the regularity assumptions on the boundary of the domain where severely reduced in [8]. A deeper knowledge of the behavior of the composition operator may lead to better numerical methods for the electric impedance tomography (see [2], for instance).

We study quasiconformal mappings ϕ whose Jacobian determinant J_ϕ satisfies the estimates

$$\left(\int_U J_\phi(x)^a\,dx\right)^{\frac{1}{a}} \leq C_a, \qquad \text{and} \qquad \left(\int_U J_\phi(x)^{-b}\,dx\right)^{\frac{1}{b}} \leq C_b \tag{1.1}$$

for values $a > 1$ and $b > 0$, where $U \subset \mathbb{R}^n$ is a certain domain (open and connected set). The existence of these values for any domain U compactly contained in Ω can be derived from quasiconformality itself, but we may have better exponents for particular mappings, and this will imply better behavior of the composition operator.

In [10, Theorems 1.1 and 1.2], it is shown that, under the first condition in (1.1) the composition operator T_ϕ sends compactly supported $W^{1,p}$ functions to $W^{1,q}$ for certain couples $n \leq q < p$, that is, some integrability of the function and its gradient is lost. The main result of the present paper shows that this loss of integrability is common to all the Bessel potential spaces $H^{s,p}$ with $0 \leq s \leq 1$ and $1 < p < \infty$ and Besov spaces $B^s_{p,r}$ with $0 < s < 1$, $1 < p < \infty$ and $0 < r \leq \infty$ (see Section 3 for the definitions). Note that $H^{0,p} = L^p$ and $H^{1,p} = W^{1,p}$.

Theorem 1.1. *Let $n \geq 2$, $0 \leq s \leq 1$ and $1 < p < \infty$. Given a quasiconformal mapping $\phi : \Omega \to \Omega'$ between two domains in $\mathbb{R}^n$, a ball B with $2B \subset \Omega$, a ball B' with $\phi(2B) \subset B'$ and positive real numbers a, b, C_a and C_b satisfying (1.1) for $U = 2B$, let q be defined by $\frac{1}{q} = \frac{1}{p} + \frac{1}{c}\left|\frac{s}{n} - \frac{1}{p}\right|$, where we take $c = a$ if $sp \geq n$ and $c = b$ if $sp < n$. If $q > 1$, then there exists a constant C such that*

$$\|T_\phi f\|_{H^{s,q}(B)} \leq C\|f\|_{H^{s,p}(B')} \tag{1.2}$$

and, if in addition $s \notin \{0, 1\}$, then

$$\|T_\phi f\|_{B^s_{q,r}(B)} \leq C\|f\|_{B^s_{p,r}(B')} \tag{1.3}$$

for every locally integrable function f and every $0 < r \leq \infty$, with constants not depending on ϕ. If $s \in \{0, 1\}$ and $p = \infty$, then (1.2) holds as well.

Previous results ([6, Proposition 4.2] and [11, Theorem 1.2]) show that T_ϕ sends compactly supported $H^{s,q} \cap L^\infty$ functions (with $0 < s < 1$) to $H^{\beta,q}$ functions for certain $\beta < s$, that is, with a loss on the smoothness parameter. More precisely, the statement of the latter theorem settles that question for mappings between diagonal Besov spaces $B^s_{q,q} \to B^\beta_{q,q}$, and it establishes the supremum of the admissible values of β as $\frac{bs}{b+1-\frac{sq}{n}} = \frac{\frac{s}{q}}{\frac{1}{q}+\frac{1}{b}\left(\frac{1}{q}-\frac{s}{n}\right)}$.

These results can be recovered from Theorem 1.1. Indeed, by [15, Theorem 2.2.5], for $0 < s < \infty$, $0 < q < \infty$, $0 < r, \ell \leq \infty$ and $0 < \Theta < 1$,

$$\|f\|_{F^{\Theta s}_{\frac{q}{\Theta},r}} \leq C_{s,q,r,\Theta}\|f\|_{F^s_{q,\ell}}\|f\|_{L^\infty} \quad \text{and} \quad \|f\|_{B^{\Theta s}_{\frac{q}{\Theta},\frac{r}{\Theta}}} \leq C_{s,q,r,\Theta}\|f\|_{B^s_{q,r}}\|f\|_{L^\infty}.$$