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Improved bounds for Fourier coefficients of Siegel modular forms

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ABSTRACT

In this paper we improve existing bounds for Siegel modular forms of small weight.

1. Introduction and statement of results

The goal of this paper is to improve existing bounds for Fourier coefficients of higher genus Siegel modular forms of small weight.

To recall what is known for genus 1, let $\Delta(\tau) := q\prod_{n=1}^{\infty}(1-q^n)^{24}$ ($q := e^{2\pi i\tau}$) be the classical Δ-function and denote by $\tau(n)$ its Fourier coefficients. The *Ramanujan conjecture* states that, for p prime,

$$|\tau(p)| \leq 2p^{\frac{11}{2}}.$$

This conjecture has been generalized for general, positive integral weight modular forms. The so-called *Ramanujan–Petersson conjecture* states that if $f(\tau) = \sum_{n=1}^{\infty} a(n)q^n$ is a weight k cusp form on a congruence subgroup, then, as $n \to \infty$,

$$a(n) \ll_{\varepsilon,f} n^{\frac{k-1}{2}+\varepsilon} \qquad (\varepsilon > 0). \tag{1.1}$$

The estimate (1.1) follows from Deligne's proof of the Weil conjectures [5,6], using highly complicated methods from algebraic geometry.

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There are many related conjectures for more complicated types of automorphic forms. In this paper, we consider the case of Siegel modular form of genus $g > 1$. For this, let F be a cusp form of weight $k \in \mathbb{N}$ with respect to the Siegel modular group $\Gamma_g := \mathrm{Sp}_g(\mathbb{Z}) \subset \mathrm{GL}_{2g}(\mathbb{Z})$ with Fourier coefficients $a(T)$, where T is a positive definite symmetric half-integral $g \times g$ matrix. Then a conjecture of Resnikoff and Saldaña [9] says that

$$a(T) \ll_{\varepsilon,F} \det(T)^{\frac{k}{2}-\frac{(g+1)}{4}+\varepsilon} \qquad (\varepsilon > 0).$$

For $g = 1$ this is exactly the Ramanujan–Petersson conjecture. For higher genus g, however, there are counterexamples coming from lifts (cf. [8]).

For $k > g + 1$, the best known estimate is

$$a(T) \ll_{\varepsilon,F} \det(T)^{\frac{k}{2}-c_g+\varepsilon} \qquad (\varepsilon > 0), \tag{1.2}$$

where

$$c_g := \begin{cases} \frac{13}{36} & \text{if } g = 2 \quad [8], \\ \frac{1}{4} & \text{if } g = 3 \quad [2], \\ \frac{1}{2g} + \left(1 - \frac{1}{g}\right)\alpha_g & \text{if } g > 3 \quad [1]. \end{cases}$$

Here

$$\alpha_g^{-1} := 4(g-1) + 4\left[\frac{g-1}{2}\right] + \frac{2}{g+2}. \tag{1.3}$$

In [3] and [4] it was shown that (1.2) still holds for $k = g + 1$ and $k = g$, respectively. Moreover, for $(g + 3)/2 < k < g$, we have [4]

$$a(T) \ll_{\varepsilon,F} \det(T)^{\frac{k}{2}-\left(1-\frac{1}{g}\right)\alpha_g+\varepsilon}. \tag{1.4}$$

In this paper we improve (1.4) and obtain

Theorem 1.1. *We have for $g/2 + 1 < k < g$*

$$a(T) \ll_{\varepsilon,F} \det(T)^{\frac{k}{2}+\frac{g-k}{2g(g-2)}-\frac{1}{2g}-\left(1-\frac{1}{g}\right)\alpha_g+\varepsilon}.$$

Remark. Theorem 1.1 is indeed an improvement since

$$\frac{g-k}{2g(g-2)} - \frac{1}{2g} < 0.$$

Our proof follows the idea of [1] using a Jacobi decomposition of Siegel modular forms. Our main achievement is an improved bound for Kloosterman sums.

The paper is organized as follows. In Section 2 we recall basic facts about Jacobi forms and their relation to Siegel modular forms. In Section 3 we bound higher dimensional Kloosterman sums. Section 4 is devoted to estimating coefficients of Poincaré series, in Section 5 we then conclude our main theorem.