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Existence and multiplicity of positive solutions for a perturbed semilinear elliptic equation with two Hardy–Sobolev critical exponents ☆

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In this paper, by applying Ekeland's variational principle and strong maximum principle, we investigate the existence and multiplicity of positive solutions for a perturbed semilinear elliptic equation with two critical Hardy–Sobolev exponents and boundary singularities.

1. Introduction and main results

In this paper, we consider the following semilinear elliptic equation

$$\begin{cases} -\Delta u - \mu \dfrac{u}{|x|^2} = \dfrac{|u|^{2^*(s_1)-2}}{|x|^{s_1}} u + \dfrac{|u|^{2^*(s_2)-2}}{|x|^{s_2}} u + \lambda f(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where Ω is an open bounded domain in R^N $(N \geq 3)$ with C^2 boundary $\partial\Omega$ and $0 \in \partial\Omega$, $0 \leq \mu < \overline{\mu} \triangleq \frac{(N-2)^2}{4}$, $0 \leq s_1, s_2 < 2$, $2^*(s_i) = \frac{2(N-s_i)}{N-2}(i = 1,2)$ are the Hardy–Sobolev critical exponents and $2^*(0) = 2^* = \frac{2N}{N-2}$ is the Sobolev critical exponent, $\lambda > 0$ is a real parameter, since we consider the existence of positive solutions of the problem (1), so we may define $f(x,t) = 0$ for $x \in \Omega$, $t \leq 0$.

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The functional corresponding to problem (1) is

$$I(u)=\frac{1}{2}\int_{\Omega}\left(|\nabla u|^2-\mu\frac{u^2}{|x|^2}\right)dx-\frac{1}{2^*(s_1)}\int_{\Omega}\frac{(u^+)^{2^*(s_1)}}{|x|^{s_1}}dx-\frac{1}{2^*(s_2)}\int_{\Omega}\frac{(u^+)^{2^*(s_2)}}{|x|^{s_2}}dx-\lambda\int_{\Omega}F(x,u^+)dx,$$

where $u\in H_0^1(\Omega), u^+=\max\{u,0\}$, $F(x,t)=\int_0^t f(x,\tau)d\tau$. It is well known that there exists a one to one correspondence between the nonnegative solutions of (1) and the critical points of I on $H_0^1(\Omega)$. More precisely we say that $u\in H_0^1(\Omega)$ is a weak solution of problem (1), if for any $v\in H_0^1(\Omega)$, there holds

$$\langle I'(u),v\rangle=\int_{\Omega}\left(\nabla u\nabla v-\mu\frac{uv}{|x|^2}\right)dx-\int_{\Omega}\frac{(u^+)^{2^*(s_1)-1}}{|x|^{s_1}}vdx-\int_{\Omega}\frac{(u^+)^{2^*(s_2)-1}}{|x|^{s_2}}vdx-\lambda\int_{\Omega}f(x,u^+)vdx=0.$$

It follows from the Hardy inequality, for $0\le\mu<\overline{\mu}$, $\|u\|:=\left(\int_\Omega\left(|\nabla u|^2-\mu\frac{u^2}{|x|^2}\right)\mathrm{d}x\right)^{1/2}$ is well defined on $H_0^1(\Omega)$ and $\|\cdot\|$ is comparable to the usual norm of $H_0^1(\Omega)$ (see[9]). However, if $\mu=\overline{\mu}$, the operator $-\Delta-(N-2)^2/(4|x|^2)$ is not equivalent to $-\Delta$ any more (see [22]). In this paper, we assume that $0\le\mu<\overline{\mu}$ and define the best Hardy–Sobolev constant by

$$V_{\mu,s_i}(\Omega)=\inf_{u\in H_0^1(\Omega)\backslash\{0\}}\frac{\int_\Omega(|\nabla u|^2-\mu\frac{u^2}{|x|^2})dx}{\left(\int_\Omega\frac{|u|^{2^*(s_i)}}{|x|^{s_i}}dx\right)^{\frac{2}{2^*(s_i)}}},i=1,2. \tag{2}$$

In the past twenty years, the singular equations with Hardy–Sobolev critical exponent or Hardy term have been studied by a large number of celebrated boffins. On the one hand, a lot of boffins work on Hardy–Sobolev critical exponent under the case of $0\in\Omega$, such as [6–8] and the references therein. On the other hand, boundary singularities problems ($0\in\partial\Omega$) have widely aroused people's interest, such as [19]. Ghoussoub with Kang [11] and Ghoussoub with Robert [12] firstly investigated the case of $0\in\partial\Omega$, such problems can be found in [4,11–13,15,18,21,25] and so on. In particular, in [11], Ghoussoub and Kang deduced that the best Hardy–Sobolev constant can be obtained in $H_0^1(\Omega)$ when $N\ge 4$ and the principle curvatures of $\partial\Omega$ are negative. Besides, if $f(x,t)=t$ in problem (4), they also proved the existence a weak solution under the assumptions $N\ge4$, $\mu=0$, $0<\lambda<\lambda_1$ (the first eigenvalue of $-\Delta$ on $H_0^1(\Omega)$) and the principle curvatures are non-positive. In [25], Yang and Chen attained two positive solutions about problem (4) with $f(x,t)=t^{q-1}$, $0<q<1$. In [26], Zhong and Zou attained a ground state solution and a positive solution with $f(x,t)=\frac{t^q}{|x|^s}$, $0\le s<2$ and $1<q\le 2^*(s)$.

In recent years, the elliptic problems with multiple Hardy–Sobolev exponents have been considered widely. For the bounded domain $\Omega\subset R^N(N\ge3)$, the following problem (3) has been studied by a lot of scholars:

$$\begin{cases}-\Delta u-\mu\dfrac{u^{q-1}}{|x|^s}=\displaystyle\sum_{j=1}^{l}\lambda_j\frac{u^{2^*(s_j)-1}}{|x|^{s_j}}+\lambda f(x,u), & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega,\end{cases} \tag{3}$$

where $\mu,\lambda_i(i=1,2,\cdots,l)$, λ are the real parameters. Li and Lin in [17] investigated (3) in the case of $\mu=0$, $l=2$, $0\le s_2<s_1\le2$, $0\ne\lambda\in R$ and $0\in\partial\Omega$. They gave the existence and nonexistence of least-energy solution. Besides, they also proved the existence and nonexistence for positive entire solutions in half space under different assumptions. In [23], Wang and Xiang studied the quasilinear elliptic problem, by an approximation argument, they obtained infinitely many solutions of (3) under the certain conditions. In [24], Wang and Yang applied an abstract theorem getting infinitely many sign-changing solutions for problem (3) with two critical Hardy–Sobolev-Maz'ya critical exponents and $\mu=0$, $\lambda f(x,u)=a(x)u$.