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Minimizers for nonconvex variational problems in the plane via convex/concave rearrangements

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ABSTRACT

Recently, A. Greco utilized convex rearrangements to present some new and interesting existence results for noncoercive functionals in the calculus of variations. Moreover, the integrands were not necessarily convex. In particular, using convex rearrangements permitted him to establish the existence of convex minimizers essentially considering the uniform convergence of the minimizing sequence of trajectories and the pointwise convergence of their derivatives. The desired lower semicontinuity property is now a consequence of Fatou's lemma. In this paper we point out that such an approach was considered in the late 1930's in a series of papers by E.J. McShane for problems satisfying the usual coercivity condition. In addition, we will update some hypotheses that McShane made by making use of a result due to T.S. Angell, concerning property (D) on the avoidance of the Lavrentiev phenomenon.

1. Introduction

In this paper we consider primarily the free problem in the calculus of variations consisting of minimizing an integral functional of the form

$$J(x) = J(a,b,x) \doteq \int_a^b f(t,x(t),\dot{x}(t))\,dt \tag{1}$$

over the class of absolutely continuous functions $x : [a,b] \to \mathbb{R}$ satisfying the fixed end conditions

$$x(a) = x_a \quad \text{and} \quad x(b) = x_b, \tag{2}$$

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where $f : [a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous. To establish existence the classical assumptions are that $z \mapsto f(t, x, z)$ is a convex function for each $(t, x) \in [a, b] \times \mathbb{R}$ and that it satisfies an appropriate coercivity or growth condition. As is well known the growth condition implies that f is bounded below either by a constant or by a Lebesgue integrable function, which immediately permits us to conclude that $J(x)$ has a finite infimum. In addition, the growth condition also allows one to conclude that any minimizing sequence has a subsequence that converges uniformly to an absolutely continuous function such that the corresponding sequence of derivatives converges weakly in the space of Lebesgue integrable functions to its derivative. That is, the minimizing sequence converges weakly in the space of absolutely continuous functions $AC[a, b]$. The convexity of $z \mapsto f(t, x, z)$ ensures that the functional J given by (1) is lower semicontinuous with respect to this topology so that any limit function of a minimizing sequence is indeed an optimal solution. The convexity is required because it is well known that weak convergence in $L^1[a, b]$ does not imply pointwise convergence (not even for a subsequence!), and that the best one can do is ensure there exists, by the Banach–Saks–Mazur theorem (see [3, 10.1.i]), a sequence of convex combinations of the derivative sequence that converges strongly in $L^1[a, b]$ to the derivative of the limit function and hence we can extract a subsequence that converges pointwise almost everywhere to the same function. Lower semicontinuity now is shown by utilizing the convexity of f in z and Fatou's lemma.

Clearly, when f does not satisfy a coercivity condition or the convexity conditions this tried and true method for proving existence is not applicable. Problems for which f does not satisfy one of the growth conditions are called noncoercive. The lack of coercivity does not allow one to conclude a minimizing sequence has a subsequence that converges weakly in $AC[a, b]$. On the other hand, the lack of convexity will not allow one to ensure that J is lower semicontinuous on $AC[a, b]$. Recently, however A. Greco [6] presented some interesting results for some noncoercive problems in which he was able to construct minimizing sequences made up of convex functions by using convex rearrangements. The topology chosen by Greco, in the space of convex functions defined on $[a, b]$ is that of uniform convergence on compact subsets. In this case it is well known that if a sequence of convex functions converges uniformly on compact subsets of $[a, b]$ its limit function is also convex and that one can extract a subsequence so that the corresponding sequence of derivatives converges pointwise almost everywhere. If it is the case that the limit function is absolutely continuous the desired lower semicontinuity property becomes a consequence of Fatou's lemma and does not require any convexity. The drawback of this approach is that the limit function need not be a continuous function. For example consider the sequence of convex functions $x_k : [-1, 1] \to \mathbb{R}$ $(k \in \mathbb{N})$ defined by $x_k(t) = |t|^k$. This sequence converges uniformly on compacta $K \subset (0, 1)$, but not on the compact interval $[0, 1]$, to the convex function

$$x_0(t) = \begin{cases} 1, & \text{if } t = \pm 1, \\ 0, & \text{if } t \in (-1, 1), \end{cases}$$

which is clearly not continuous. To show the limit function is absolutely continuous Greco then resorts to some ad hoc arguments, such as appealing to the necessary conditions (the DuBois-Reymond equation) or some additional properties of the integrand f. This last part clearly limits the applicability of this approach. On the other hand, if one could ensure the limit function is absolutely continuous, for example by coercivity, then this approach could perhaps be used to investigate existence when the convexity condition fails.

The question of whether one could use minimizing sequences of absolutely continuous functions that converge uniformly to a limit function and for which the derivative sequence converges pointwise almost everywhere to establish existence was asked much earlier than Greco's recent results. Apparently the first paper along these lines was due to H. Lewy [8] in 1928 who looked at finite difference approximations to study existence. Shortly thereafter, E.J. McShane [9,10,12–14], beginning in 1938, published a series of papers that also explore these ideas for both free problems as well as isoperimetric problems. In particular, McShane's papers [9,10] investigated convex/concave rearrangements in the same spirit as Greco.

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