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An optimal result for sampling density in shift-invariant spaces generated by Meyer scaling function

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ABSTRACT

For a class of continuously differentiable function ϕ satisfying certain decay conditions, it is shown that if the maximum gap $\delta := \sup_i (x_{i+1} - x_i)$ between the consecutive sample points is smaller than a certain number B_0, then any $f \in V(\phi)$ can be reconstructed uniquely and stably. As a consequence of this result, it is shown that if $\delta < 1$, then $\{x_i : i \in \mathbb{Z}\}$ is a stable set of sampling for $V(\phi)$ with respect to the weight $\{w_i : i \in \mathbb{Z}\}$, where $w_i = (x_{i+1} - x_{i-1})/2$ and ϕ is the scaling function associated with Meyer wavelet. Further, the maximum gap condition $\delta < 1$ is sharp.

1. Introduction

In [8], Gröchenig proved that if $\{x_i : i \in \mathbb{Z}\}$ is a sample set with $\sup_i (x_{i+1} - x_i) < 1$, then $\{x_i\}$ is a stable set of sampling for $V(\text{sinc})$ with respect to certain weight $\{w_i : i \in \mathbb{Z}\}$. In [2], Aldroubi and Gröchenig proved that if $X \subset \mathbb{R}$ is a separated set such that $\sup_i (x_{i+1} - x_i) < 1$, then any function in a shift-invariant space with B-spline as a generator can be reconstructed stably and uniquely from its samples $\{f(x_i) : x_i \in X\}$. In that paper, they conjectured that the theorem remains true for a much larger class of shift-invariant spaces. Since their methods use special properties of spline functions, they mentioned that it is not clear how to extend their result to shift-invariant spaces with even a compactly supported generator. Recently in [10], it is shown that for a class of totally positive functions ϕ of finite type, if $\delta := \sup_i (x_{i+1} - x_i) < h$, then $\{x_i : i \in \mathbb{Z}\}$ is a stable set of sampling for $V_h(\phi)$, where $V_h(\phi) = \overline{\text{span}}\{\phi(\cdot - hk) : k \in \mathbb{Z}\}$.

For sampling in shift-invariant spaces, the oscillation method dates back to the work of [1] where in they used the oscillation function $osc_\delta s$, defined by $(osc_\delta s)(x) = \sup_{|y| \le \delta} |s(x) - s(x+y)|$ in order to ob-

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tain a reconstruction method for spline-like spaces. In [12], an average sampling theorem was given for shift-invariant spaces with equally spaced sample points and arbitrary averaging functions. In [13], average sampling theorems were studied for spline subspaces with standard averaging functions $\chi_{[x_k-1/2,x_k+1/2]}$. In [14], an average sampling theorem was presented for shift-invariant spline spaces along with the optimal upper bound for the support length of averaging functions. In [16], explicit bound expression for sampling inequalities was obtained for shift-invariant spline spaces.

In this paper, we consider a class of continuously differentiable functions satisfying certain decay assumptions. We also assume that $\{T_n\phi : n \in \mathbb{Z}\}$ forms a Riesz basis for $V(\phi)$ and $\operatorname{esssup}_{w\in[0,1]} \sum_{l\in\mathbb{Z}}(w+l)^2|\widehat{\phi}(w+l)|^2 < \infty$. Then we show that if $\sup_i(x_{i+1} - x_i)$ is smaller than a certain number B_0, then $\{x_i : i \in \mathbb{Z}\}$ is a stable set of sampling for $V(\phi)$ with respect to the weight $\{w_i\}$, where $w_i = (x_{i+1} - x_{i-1})/2$. In order to prove the above result, first we prove a Bernstein-type inequality, namely, $\|f'\|_2 \leq 2\pi\sqrt{B}\|f\|_2$, for every $f \in V(\phi)$. This helps us to get the required bound B_0 as $\frac{1}{2\sqrt{B}}$, towards the maximum gap condition. As a consequence of sampling theorem for Meyer scaling function ϕ, we show that if $\delta < 1$, then $\{x_i : i \in \mathbb{Z}\}$ is a stable set of sampling for $V(\phi)$ with respect to the weight $\{w_i : i \in \mathbb{Z}\}$. Further, we show that the maximum gap condition $\delta < 1$ is sharp. We wish to emphasize that many of the earlier papers with explicit sampling bound conditions available in the literature were far away from "sharpness" results and up to our knowledge, the present "sharp" result could not be obtained from the available conditions in the literature.

Further, we notice that in the case of Bernstein's inequality for $V(\phi)$, it is not always possible to find the exact value of B except may be in the case of functions whose Fourier transform has compact support. So finally, we show that if ϕ is a differentiable function with support $[a, b]$, then one can explicitly calculate a bound instead of B. In due course, we also provide a sampling formula for reconstructing a function f belonging to $V(\phi)$, where ϕ satisfies the above condition, from its nonuniform samples. We refer to a recent paper of the authors [4], where an explicit sampling formula using complex analysis technique is provided for reconstructing a function f belonging to $V(\phi)$, where ϕ is a compactly supported even function, from its uniform samples.

2. Notations and background

Definition 2.1. A sequence of vectors $\{f_n : n \in \mathbb{Z}\}$ in a separable Hilbert space $\mathcal{H}$ is said to be a *Riesz basis* if $\overline{span\{f_n\}} = \mathcal{H}$ and there exist constants $0 < c \leq C < \infty$ such that

$$c\sum_{n\in\mathbb{Z}}|d_n|^2 \leq \Big\|\sum_{n\in\mathbb{Z}} d_n f_n\Big\|^2_{\mathcal{H}} \leq C\sum_{n\in\mathbb{Z}}|d_n|^2, \tag{2.1}$$

for all $(d_n) \in \ell^2(\mathbb{Z})$. Equivalently, a Riesz basis is an image of an orthonormal basis under a bounded invertible operator.

Definition 2.2. A sequence of vectors $\{f_n : n \in \mathbb{Z}\}$ in a separable Hilbert space $\mathcal{H}$ is said to be a *frame* if there exist constants $0 < A \leq B < \infty$ such that

$$A\|f\|^2_{\mathcal{H}} \leq \sum_{n\in\mathbb{Z}}|\langle f, f_n\rangle_{\mathcal{H}}|^2 \leq B\|f\|^2_{\mathcal{H}}, \tag{2.2}$$

for every $f \in \mathcal{H}$.