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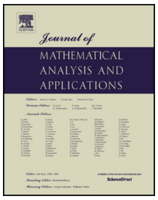

On local solvability of a class of abstract underdetermined systems

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ABSTRACT

In this work we present a necessary and sufficient condition for a class of abstract underdetermined systems to be solvable. We develop J. F. Trèves' ideas, presenting the so called condition (ψ) and its connection with the study of the solvability in consideration. We also prove the existence of finite order regularity solutions.

1. Introduction

In this paper we study the local solvability in top degree of the differential complex defined by the operators

$$\mathrm{L}_j = \partial_{t_j} - (\partial_{t_j}\phi)(t, A)A, \quad j = 1, \dots, n,$$

where A is a linear operator, densely defined in a Hilbert space H. We shall assume that A is unbounded, but it is self-adjoint, *positive definite* and it has a bounded inverse A^{-1}; and where $\phi(t, A)$ are power series with respect to A^{-1}, with coefficients in $C^\infty(\Omega)$, for some open set $\Omega \subset \mathbb{R}^n$, that is,

$$\phi(t, A) = \sum_{k \geq 0} \phi_k(t) A^{-k}.$$

These power series are assumed to be convergent in $L(H, H)$, as well as each of their t-derivatives, uniformly with respect to t on compact subsets of Ω.

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Our analysis will focus on a neighborhood Ω of the origin. If $\Omega \subset \mathbb{R}$ is an open interval containing the origin, [8] shows us necessary and sufficient conditions for the local solvability and hypoellipticity of the operator $\mathrm{L} = \partial_t - \phi(t, A)A$ at $t = 0$.

This work concerns the following problem:

For each $k \in \mathbb{Z}_+$ such that $N > \frac{n+k}{2}$, $N \in \mathbb{Z}_+$, find open neighborhoods of 0, $\omega_N \subset \Omega_N$, such that

$$\forall f \in C^\infty_{(n)}(\Omega_N, H^\infty),\ \exists v^{(N)} \in C^k_{(n-1)}(\omega_N, H^k) \text{ such that } \mathbb{L}v^{(N)} = f \ \text{ in } \ w_N, \tag{1.1}$$

where

$$\mathbb{L}v^{(N)} \ = \ \Big(\sum_{j=1}^{n} \mathrm{L}_j v_j^{(N)}\Big) dt_1 \wedge \ldots \wedge dt_n.$$

We denote by $C^\omega(\Omega)$ the space of analytic functions in Ω and assume $\phi_0 \in C^\omega(\Omega)$. In the text, $\Re\phi_0$ and $\Im\phi_0$ denote the real part and the imaginary part of ϕ_0, respectively.

Let B be the ball $\{t \in \mathbb{R}^n : |t| < R\} \subset\subset \Omega$.

Definition 1.1. We say that condition (ψ_1) holds on B if, for every real number a, the set

$$\{t \in B : \Re\phi_0 \leq a\} \text{ has no compact connected components.}$$

Definition 1.2. We say that condition (ψ_2) holds on B if, for every real number a, the set

$$\{t \in B : \Re\phi_0 \geq a\} \text{ has no compact connected components.}$$

Definition 1.3. We say that conditions (ψ_1) and (ψ_2) hold at 0 if, for any open ball B centered at the origin, there exists an open subset $\Omega' \subset B$ containing 0 such that both (ψ_1) and (ψ_2) hold on Ω'.

The main result states:

Theorem 1.4. *Condition (ψ_1) at 0 is necessary and sufficient to solve (1.1).*

The case where A is a linear operator, densely defined in H, unbounded and self-adjoint is also considered. That is, we study the problem:

For each $k \in \mathbb{Z}_+$ such that $N > \frac{n+k}{2}$, $N \in \mathbb{Z}_+$, find open neighborhoods of 0, $\omega_N \subset \Omega_N$, such that

$$\forall f \in C^\infty_{(n)}(\Omega_N, H^\infty),\ \exists v^{(N)} \in C^k_{(n-1)}(\omega_N, H^k) \text{ such that } \mathbb{L}_0 v^{(N)} = f \ \text{ in } \ w_N, \tag{1.2}$$

where

$$\mathbb{L}_0 v^{(N)} \ = \ \Big(\sum_{j=1}^{n} \mathrm{L}_{j,0} v_j^{(N)}\Big) dt_1 \wedge \ldots \wedge dt_n, \qquad \mathrm{L}_{j,0} = \partial_{t_j} - (\partial_{t_j}\Re\phi_0)(t)A, \quad j = 1, \ldots, n.$$

The result proved is the following:

Theorem 1.5. *Conditions (ψ_1) and (ψ_2) at 0 are necessary and sufficient to solve (1.2).*

Remark. In problems (1.1) and (1.2) a category argument shows that ω_N can be assumed to depend only on Ω_N and not on f.