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Topological invariants and Lipschitz equivalence of fractal squares

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ABSTRACT

Fractal sets typically have very complex geometric structures, and a fundamental problem in fractal geometry is to characterize how "similar" different fractal sets are. The Lipschitz equivalence of fractal sets is often used to classify fractal sets that are geometrically similar. Interesting links between Lipschitz equivalence and algebraic properties of contraction ratios for self-similar sets have been uncovered and widely analyzed. However, with the exception of very few papers, the study of Lipschitz equivalence has largely focused on totally disconnected self-similar sets. For connected self-similar sets this problem becomes rather challenging, even for well known fractal models such as fractal squares. In this paper, we introduce geometric and topological methods to study the Lipschitz equivalence of connected fractal squares. In particular we completely characterize the Lipschitz equivalence of fractal squares of order 3 in which one or two squares are removed. We also discuss the Lipschitz equivalence of fractal squares of more general orders. Our paper is the first study of Lipschitz equivalence for nontrivial connected self-similar sets, and it raises also some interesting questions for the more general setting.

1. Introduction

The study of "similarities" among different objects is a fundamental problem in several areas of mathematics, most notably in geometry and topology. The problem of measuring similarities or differences among fractal sets has played a central role in fractal geometry. The concept of dimension, whether it is the Hausdorff dimension or the box counting dimension, is widely used for such a purpose.

Two compact sets, even with the same dimension, may in fact be quite different in many ways. A stronger measure of similarity in fractal geometry is the notion of *Lipschitz equivalence*. We say two metric spaces $(X; d_X)$ and $(Y; d_Y)$ are Lipschitz equivalent if there exists a bi-Lipschitz map f from X to Y, that is, f is a bijection and there exist constants $c_1, c_2 > 0$ such that

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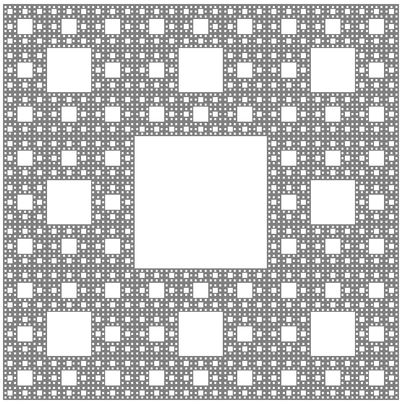

Fig. 1. The classical Sierpinski carpet.

$$c_1 d_X(x_1, x_2) \le d_Y(f(x_1), f(x_2)) \le c_2 d_X(x_1, x_2)$$

for all $x_1, x_2 \in X$.

The Lipschitz equivalence for self-similar fractal sets has gained some attention in recent years. There are two distinct flavors among the studies. In one direction a strong focus is placed on finding connections between Lipschitz equivalence of self-similar sets to the algebraic properties of the contraction ratios of the sets. One of the very first results in this area was obtained in Falconer and Marsh [7], where the authors have given necessary conditions for two dust-like self-similar sets to be Lipschitz equivalent. This direction was further exploited in Rao, Ruan and Wang [18], in which the authors obtained a complete classification of Lipschitz equivalence for dust-like self-similar sets in $\mathbb{R}^n$ generated from two similitudes. Other advances in this direction can be found in Cooper and Pignataro [3], David and Semmes [4], Fan, Rao and Zhang [8], Mattila and Saaranen [16], Rao and Zhang [24], Xi [29], and Xi and Xiong [32]. Another direction focuses more on the geometric properties of the maps and how they affect Lipschitz equivalence. A good example in this direction is the $\{1,3,5\} - \{1,4,5\}$ problem posed by David and Semmes in [4]. This question was answered in affirmative in Rao, Ruan and Xi [20], and later generalized in [30] and [26] in one dimensional case. It was also generalized by several researchers in higher dimensional case, including Roinestad [25], and Xi and Xiong [31]. There are also works in other directions. For examples, Luo and Lau [14] studied Lipschitz equivalence via hyperbolic boundaries, and Rao, Ruan and Yang [21] studied Lipschitz equivalence via gap sequence. For other relative works, please see [5,6,10,19].

The majority of the aforementioned work study the Lipschitz equivalence only for totally disconnected self-similar sets. Results on Lipschitz equivalence of self-similar sets that are not totally disconnected are few and far between. Several studies focused on a special class of fractal sets called fractal squares. Let $N \ge 2$. A *fractal square* K *of order* N is a nonempty compact set in $\mathbb{R}^2$ satisfying $K = \frac{1}{N}(K + \mathcal{D})$ where $\mathcal{D}$ is a nonempty proper subset of $\mathbb{Z}_N^2$ where $\mathbb{Z}_N := \{0, 1, \dots, N-1\}$. For example, the classical Sierpinski carpet is $K(3, \mathcal{D})$, where the complement of $\mathcal{D}$ is the middle piece, i.e. $\mathcal{D}^c = \{(1,1)\}$. See Fig. 1. Lau, Luo and Rao [13] studied topological structures of fractal squares, and they proved that K has a nontrivial connected component that is *not* a line segment if and only if the complement of $K + \mathbb{Z}^2$ has a bounded connected component. Wen, Zhu and Deng [27] obtained a complete classification of Lipschitz equivalence of fractal squares of order 3 with $|\mathcal{D}| = 4$. Under this setting, either a fractal square K is totally disconnected, or all non-trivial components of K are parallel line segments. It is proved that there are exactly two different Lipschitz equivalent classes. The Lipschitz equivalence of the totally disconnected cases there follows from [31]. For the non-totally disconnected cases one can in fact explicitly write down the bi-Lipschitz maps. Recently, based on the results of [13] and [31], Luo and Liu [15] presented partial classification of fractal squares of order 3 and $|\mathcal{D}| = 5$.