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Essential spectrum of non-self-adjoint singular matrix differential operators

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ABSTRACT

The purpose of this paper is to study the essential spectrum of non-self-adjoint singular matrix differential operators in the Hilbert space $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ induced by matrix differential expressions of the form

$$\begin{pmatrix} \tau_{11}(\,\cdot\,, D) & \tau_{12}(\,\cdot\,, D) \\ \tau_{21}(\,\cdot\,, D) & \tau_{22}(\,\cdot\,, D) \end{pmatrix}, \tag{0.1}$$

where τ_{11}, τ_{12}, τ_{21}, τ_{22} are respectively m-th, n-th, k-th and 0 order ordinary differential expressions with $m = n + k$ being even. Under suitable assumptions on their coefficients, we establish an analytic description of the essential spectrum. It turns out that the points of the essential spectrum either have a local origin, which can be traced to points where the ellipticity in the sense of Douglis and Nirenberg breaks down, or they are caused by singularity at infinity.

1. Introduction

The spectral analysis of singular matrix differential operators generated by matrix differential expressions as in (0.1) is important in many branches of theoretical physics including magnetohydrodynamics and astrophysics, see e.g. [22,3]. For example, in models of linear stability theory of plasmas confined to a toroidal region in $\mathbb{R}^3$, by eliminating one variable by means of the S^1-symmetry, one arrives at second order systems of partial differential equations in the radial and angular variables on the cross section of the torus. Using a Fourier series decomposition with respect to the angular variable, the operator matrix then becomes a direct sum of operators of the form (0.1), see e.g. [4,8]. In view of linear stability analysis and numerical approximations, it is of crucial importance to have information on the location of the whole essential spectrum of operator matrices as in (0.1).

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Two of the reasons why such matrix differential operators received continuous attention of specialists in spectral theory during the last thirty years may be explained as follows. Firstly, in contrast to the case of scalar differential operators, matrix differential operators need not to have empty essential spectrum even if the underlying domain is compact and the corresponding boundary conditions are regular. This is due to the matrix structure which allows for essential spectrum to arise because of the violation of ellipticity in an appropriate sense. Secondly and most interestingly, in the case when the underlying domain is not compact, the essential spectrum of the matrix differential operator cannot be approximated by the essential spectra of operators determined by the same operator matrix on an increasing sequence of compact sub-domains exhausting to the original domain. In fact, it turns out that the essential spectrum can have a branch caused because of the singularity at infinity or, more generally, at the boundary of a non-compact interval.

Essential spectrum of matrix differential operators generated by (0.1) is well known if the underlying domain is compact, see [2]. The situation is much more complicated if the underlying domain is non-compact. In this case the spectral properties are far from being fully-understood up to date in the non-self-adjoint setting, especially when the matrix differential operator is not a perturbation of a self-adjoint operator.

The appearance of the branch of essential spectrum due to the singularity at the boundary was first predicted by Descloux and Geymonat [4] in connection with a physical model describing the oscillations of plasma in an equilibrium configuration in a cylindrical domain and proven much later by Faierman, Mennicken and Möller [8]. Similar phenomena in connection with problems of theoretical physics have been studied by many authors including Kako [14–16], Descloux and Kako [17], Raikov [29,30], Beyer [3], Atkinson, H. Langer, Mennicken and Shkalikov [2], H. Langer and Möller [21], Faierman, Mennicken and Möller [6,7], Hardt, Mennicken and Naboko [10], Konstantinov [18], Mennicken, Naboko and Tretter [24], Kurasov and Naboko [20], Möller [26], Marletta and Tretter [23], Kurasov, Lelyavin and Naboko [19], Qi and Chen [27,28].

Most of these studies were concerned with the investigation of particular and "almost-symmetric" operators and it was shown that the essential spectrum due to the singularity at the boundary appears because of a very special interplay between the matrix entries. The first analytic description the essential spectrum in the general setting was established in [11] in the symmetric case with $m = 2$, $n = k = 1$ and $[0, \infty)$ instead of $\mathbb{R}$. The results of [11] were later extended to much wider classes of symmetric matrix differential operators under considerably weaker assumptions in [12] where the second diagonal entry allowed to be a matrix multiplication operator.

The current manuscript seems to be the first attempt to investigate the essential spectrum, in particular, both above mentioned spectral phenomena, for *non-self-adjoint* matrices of ordinary differential operators of mixed-orders on the real line. The aim is to establish an analytic description of the entire essential spectrum in terms of the coefficients of (0.1). Our method to describe the part of the essential spectrum caused by the singularity at infinity analytically is different from the so-called "cleaning of the resolvent" approach suggested in [20], which was based on a result on the essential spectrum of separable sum of pseudo-differential operators, see [20, Theorem A.1]. Nevertheless, the remarks in [20] concerning the non-self-adjoint case have inspired the present paper.

The paper is organized as follows. Section 2 provides the necessary operator theoretic framework for the matrix differential operator generated by (0.1) in the Hilbert space $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ as well as for its first Schur complement. Section 3 is dedicated to the description of the essential spectrum due to the singularity at infinity. It is characterized in terms of the essential spectrum of the first Schur complement using the characterization of Fredholm operators in terms of approximate/generalized inverses and pseudo-differential operator techniques. In Section 4 the essential spectrum due to the violation of ellipticity in the sense of Douglis and Nirenberg is described. Section 5 contains the main result of the paper (see Theorem 5.3), where by suitable gluing and smoothing the results of the two previous sections are blended together. The obtained analytic description of the whole essential spectrum is given in terms of the original coefficients of the matrix differential operator in (0.1) and illustrated by an example.