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The cost of the control in the case of a minimal time of control: The example of the one-dimensional heat equation

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A B S T R A C T

In this article, we consider the controllability of the one-dimensional heat equation with Dirichlet boundary conditions, internal control depending only on the time variable and an imposed profile depending on the space variable. It is well-known that in this context, there might exist a positive minimal time of null-controllability T_0, depending on the behavior of the Fourier coefficients of the profile. We prove two different results. The first one, which is surprising, is that the cost of the controllability in time $T > T_0$ close to T_0 may explode in an arbitrary way. On the other hand, we prove as a second result that for a large class of profiles, the cost of controllability at time $T > T_0$ is bounded from above by $\exp(C(T_0)/(T-T_0))$ for some constant $C(T_0) > 0$ depending on T_0. The main method used here is the moment method.

1. Introduction

Let $T > 0$. In what follows, we will consider the following controlled heat equation on $(0,T) \times (0,\pi)$, with Dirichlet boundary conditions:

$$\begin{cases} y_t - y_{xx} = f(x)u(t) & \text{in } (0,T) \times (0,\pi), \\ y(0,\cdot) = y^0 & \text{in } (0,\pi), \\ y(\cdot,0) = y(\cdot,L) = 0 & \text{in } (0,T), \end{cases} \tag{1}$$

where $y^0 \in L^2(0,\pi)$, $u \in L^2(0,T)$ is the control and $f \in H^{-1}(0,\pi)$ is an imposed profile for this control.

It is well-known that equation (1) is well-posed in the sense that there exists a unique solution $y \in C^0([0,T], L^2(0,\pi)) \cap L^2((0,T), H_0^1(0,\pi))$ verifying moreover that there exists a constant $C > 0$ such that for every $y^0 \in L^2(0,\pi)$, every $f \in H^{-1}(0,\pi)$ and every $v \in L^2(0,T)$, we have

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$$||y||_{C^0([0,T],L^2(0,\pi))} + ||y||_{L^2((0,T),H_0^1(0,\pi))} \leqslant C(||y^0||_{L^2(0,\pi)} + ||f||_{H^{-1}(0,\pi)}||v||_{L^2(0,T)}),$$

which implies in particular that the control operator $u \in \mathbb{R} \mapsto f(\cdot)u$ is admissible for the semigroup $e^{t\Delta}$ with domain $D(\Delta)$. The controllability properties of this equation have been widely studied (see for instance [11,8,2]). The approximate controllability can be easily characterized by the condition

$$f_k \neq 0, \forall k \in \mathbb{N}^*, \tag{2}$$

where

$$f_k := \langle f, e_k \rangle_{H^{-1}(0,\pi),H_0^1(0,\pi)}.$$

Assume from now on and until the end of the article that the condition (2) is satisfied. Concerning the study of the null-controllability of (1), one very efficient tool in the one-dimensional case is the celebrated *moment method* introduced in [10]. Let us present quickly this method. We consider the 1-D Laplace operator ∂_{xx} with domain $D(\Delta) := H^2(0,\pi) \cap H_0^1(0,\pi)$ and state space $H := L^2(0,\pi)$. It is well-known that $-\Delta : D(\Delta) \to L^2(0,\pi)$ is a positive definite operator with compact resolvent, the k-th eigenvalue is $\lambda_k = k^2$, an associated normalized eigenvector is

$$e_k(x) := \frac{\sqrt{2}}{\sqrt{\pi}} \sin(kx).$$

We decompose the initial condition y^0 on the Hilbert basis e_k:

$$y^0(x) = \sum_{k=1}^{\infty} a_k e_k(x), \tag{3}$$

where $(a_k)_{k\in\mathbb{N}^*} \in l^2(\mathbb{N}^*)$. Then, it is classical that imposing $y(T,\cdot) = 0$ is equivalent to imposing

$$\int_0^T e^{\lambda_k t} u(t)dt = -\frac{a_k}{f_k}, \ \forall k \in \mathbb{N}^*. \tag{4}$$

Hence, u needs to be the solution of a moment problem which can be solved by finding a *biorthogonal family* to the family of exponentials $\{\exp(\lambda_k t)\}_{k\in\mathbb{N}^*}$ on $(0,T)$. We introduce the following quantities:

$$I_k(f) := -\frac{\log(|f_k|)}{k^2} \tag{5}$$

and

$$T_0 := \limsup_{k\to\infty} I_k(f) \in [0,\infty]. \tag{6}$$

It is proved in [2] that:

1. System (1) is null-controllable at any time $T > T_0$.
2. System (1) is not null-controllable at any time $T < T_0$.

Hence, there might exist a positive minimal time of controllability, depending on the action of the control through the profile f.