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Minimal invariant closed sets of set-valued semiflows

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ABSTRACT

In the paper we deal with minimal closed subsets invariant with respect to set-valued semiflow. Such sets are known as supports of invariant or even ergodic measures of stochastic processes associated with such semiflows. Our motivation comes from some earlier and recent results connected with bounded noise processes, but we work in the framework of set-valued semiflows with lower semicontinuous members on general metric space rather than mostly studied by many authors continuous and compact-valued ones. Such semiflows appear naturally when nonautonomous/random dynamical systems are considered.

1. Introduction

In the series of papers [10–14] we have remodeled step by step the dynamics of iterated function systems (see, for example, [4]), random as well as nonautonomous dynamical systems (see, for example, [2] and [18], for general notions and basic facts) by the dynamics of induced semiflows of associated multifunctions (called *state multifunctions*). We have considered some types of attractors as well as semiattractors by both topological and probabilistic point of view. Our first fundamental observation is that the state multifunctions are, in general, only lower semicontinuous and often closed-valued but not necessarily continuous and compact-valued as considered earlier in many standard contexts (see, for example, [5] for iterated function systems, [20] for dynamical system perturbed by some noise, [26] with random difference equations, [27] in the context of closed relations, and also [3] and [29] for set-valued semiflows generated by differential inclusions). The second observation is that if multifunctions have not necessary closed values strict invariance of attracting sets fails, but some weaken kind of invariance (called here *cl-invariance*, see Section 4 below) is still valid. Moreover, we have showed that semiflows of state multifunctions always can be considered as supports of transition probabilities generating some family or even a semigroup of Markov–Feller opera-

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tors on measures. Obtained for semiflows of state multifunctions attractors and semiattractors are strongly connected with supports of attracting measures for families of these operators (see [11, Sect. 6] and [12, Sect. 7], also [13]).

Our present development is mostly inspired by the papers [20] and [15] where minimal invariant sets with respect to the semiflow of multifunctions induced by perturbed by bounded noise dynamical system were considered from topological [20] as well as probabilistic point of view [15]. Notice now only that studying stochastic perturbation of discrete as well as continuous time dynamical systems has yet long history (see, for example, [21] and the references therein, also the references in [20]). The bounded noise case has also many applications in physics, biology and engineering (see [9]). Our results are parallel to those from [20] and [15] in the context of semiflows of lower semicontinuous multifunctions. Although our approach is very general we believe that the results could be applied in mentioned at the beginning realities as unifying earlier dispersed partial ones. In particular they can be used when assumptions on compactness of a state space or values of multivalued process are too strong. Mark here that there are stochastic processes generated, for example, by iterated function systems having even a unique stationary measure with non-compact supports (see [22], also [23] and [24]). Equivalently induced set-valued semiflow has non-compact semiattractor [12]. We also foresee that obtained here results could be applied in finding solutions of some integral–functional equations associated with random iterations (see, for example, [6,16], and the references therein). The author starts already with development in this direction. Notice that recently some results on minimal invariant sets for single multifunctions as well as iterated function systems were obtained in [25] using Knaster–Tarski fixed point principle.

We consequently use the apparatus of topological (Kuratowski's) limit instead of standard Hausdorff's metric. Such a point of view is much more general and less complicated. Moreover, we can omit many standard problems appearing when considered sets are non-compact.

The paper is organized as follows. Next two sections contain some preliminary results on topological convergence of nets of subsets of a metric space and lower semicontinuous (l.s.c.) multifunctions. In Section 4 we define semiflows of l.s.c. multifunctions and show invariance properties of theirs ω-limit sets (Theorem 4.2 below). We also characterize ω-limit sets by intersections of families of closed absorbing sets (Proposition 4.3, cf. also related results in [27]). Section 5 is devoted to minimal closed invariant sets with respect to semiflows of l.s.c. multifunctions. We prove that, under desirable, but still quite general and natural assumptions, such a set is characterized simply as ω-limit set of any its point, and consequently it consists only of recurrent points. In Section 6 we consider lower semicontinuous dependence of minimal closed invariant sets on parameters. We obtain results related to those in [20] describing topological bifurcation scenario. One can observe that we still do not need to assume continuity of the system and we assume that only minimal sets contains non-degenerated open balls (cf. [20, Sect. 5]). In [12] we have shown under suitable assumptions that with every semiflow of l.s.c. set-valued mappings one can associate a semigroup of Markov–Feller operators on the space of probabilistic Borel measures and its asymptotic properties can be 'translated' into behavior of main multivalued system, in particular support of a unique invariant and attractive probabilistic measure is the semiattractor. Here, going farther this path, in Section 7, we show that minimal invariant closed sets are no more no less, but supports of invariant or even ergodic measures of associated Markov–Feller semigroup. The results generalize those in [15] where particular case of differential equations with bounded noise on compact smooth finite dimensional manifold was considered (see also [32]).

2. Topological limits

Let (X, ϱ) be a metric space. By $B^o(x, \varepsilon)$ we denote the open ball with center x and radius ε, cl$A$ stays for a closure of A and int$A$ for its interior. As a *neighborhood* of a non-void subset B of X we mean a set $D \subset X$ such that $B \subset \text{int}D$. If D is closed it is called a closed neighborhood. A family of all neighborhoods of a subset B of X is denoted by $\mathfrak{N}(B)$. For an arbitrary $x \in X$ and a subset B of X we denote by

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