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# The symplectic non-squeezing properties of mass subcritical Hartree equations

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ABSTRACT

We prove the non-squeezing properties of all mass subcritical Hartree equations on $\mathbb{R}^d$ $(d \geq 2)$. The result is achieved by elaborating and generalizing some of the techniques in [13,14] by Killip–Visan–Zhang, and two major ingredients are the weak well-posedness for a sequence of changing equations and approximation to an infinite dimensional problem by finite dimensional problem.



## 1. Introduction

Let $0 < \lambda < 2$; we consider the mass subcritical Hartree equation on $\mathbb{R}^d$ $(d \geq 2)$

$$(i\partial_t + \Delta)u = F(u), \tag{1}$$

where $F(u) = \pm(|x|^{-\lambda} * |u|^2)u$. In this paper, we also discuss variants of (1) where the non-linearity is slightly changed:

$$(i\partial_t + \Delta)u = \alpha^4 \mathcal{P}[F(\mathcal{P}u)]. \tag{2}$$

Here $0 \leq \alpha \leq 1$, $\mathcal{P}$ is a Mikhlin multiplier with real symmetric symbol. Note that when $\alpha = 0$, (2) is the linear Schrödinger equation; when $\alpha = 1$ and $\mathcal{P} = I$, (2) becomes (1); when $\alpha = 1$ and $\mathcal{P} = P_{\leq N_n}$, we get

$$(i\partial_t + \Delta)u = P_{\leq N_n}[F(P_{\leq N_n}u)], \tag{3}$$

where $P_{\leq N_n}$ is the Littlewood–Paley projection defined in Section 2.1.

The equation (2) preserves mass and energy:

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$$M(u(t)) = \int_{\mathbb{R}^d} |u(t,x)|^2 dx,$$

$$E(u(t)) = \int_{\mathbb{R}^d} \frac{|\nabla u(t,x)|^2}{2} \pm \frac{\alpha^4}{4} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|\mathcal{P}u(t,x)|^2 |\mathcal{P}u(t,y)|^2}{|x-y|^\lambda} dxdy.$$

And the Hamiltonian formulation with respect to $E(u(t))$ is $\frac{\partial u}{\partial t} = \frac{i\partial E(u)}{\partial \bar{u}}$, see [2].

The symplectic non-squeezing theorem of Gromov on finite dimensions in [7] states that:

**Theorem 1.1** *(Gromov, [7]). Fix $0 < r_1 < R < \infty$ and $\alpha \in \mathbb{C}$. Let $B(z_*, R)$ be the ball of radius $R$ centered at $z_* \in \mathbb{C}^n$ and $\ell \in \mathbb{C}^n$ with $|l| = 1$, and suppose $\phi : B(z_*, R) \to \mathbb{C}^n$ is a symplectic map. Then there exists $z \in B(z_*, R)$ so that*

$$|\langle \ell, \phi(z)\rangle - \alpha| > r_1.$$

A symplectic map by definition is a diffeomorphism that preserves a symplectic structure. For a review of the symplectic structures, Hamiltonian mechanics, see [13] and the references therein. For symplectic non-squeezing results of BBM, KdV, Klein–Gordon on torus and other types of PDEs, see [5,8,15,17,21], which are also extensively reviewed in [13].

In [13,14], Killip, Visan and Zhang prove the non-squeezing properties of the cubic nonlinear Schrödinger equation on $\mathbb{R}^2$ and $\mathbb{R}$, which are the first symplectic non-squeezing result for Hamiltonian PDEs in infinite volume. Stimulated by their work, we prove the following

**Theorem 1.2** *(Main theorem). Fix $z_*$, $l \in L^2(\mathbb{R}^d)$ with $\|l\|_2 = 1$, $\alpha \in \mathbb{C}$, $0 < r_1 < R < \infty$ and $T > 0$. Then there exists $u_{\infty,0} \in B(z_*, R)$ such that the solution $u_\infty$ to (1) with initial data $u_{\infty,0}$ satisfies*

$$|\langle l, u_\infty(T)\rangle - \alpha| > r_1. \tag{4}$$

Two main ingredients to achieve the symplectic non-squeezing result for Hartree equation on $\mathbb{R}^d$ $(d \geq 2)$ are the weak well-posedness for a sequence of changing equations and approximation to an infinite dimensional problem by finite dimensional problem.

The standard well-posedness result in the strong $L^2(\mathbb{R}^d)$ or $H^1(\mathbb{R}^d)$ topology has been systemically studied in earlier literature (see [4,10,20,19] for instance). In [1], Bahouri and Gérard prove the weak well-posedness of the energy critical wave equation via a nonlinear profile decomposition. Namely, if the sequence of data converges weakly in $H^1(\mathbb{R}^d)$, so does the solutions to energy critical NLW on any finite time interval. One thing we may notice is that these results regarding the strong or weak well-posedness are only for a fixed single equation.

Extending the result in [1] to a single Hartree equation is not challenging, while having the weak well-posedness for a fixed equation is not sufficient. As a matter of fact, in order to make a connection with the finite dimensional problem, we need to project the solution on finite frequencies. Therefore, this requires the weak well-posedness for a sequence of changing equations (3) with the frequency $N_n \to \infty$. Obviously, introducing this changing factor to the scheme of [1] brings a lot of technical difficulties; for example, the limiting nonlinear profile will no longer always come from the solution of the same equation. The choice of the limiting equation depends crucially on the limiting behavior of several parameters: the scaling, the translation, and frequency modulation. Such difficulties were first considered in the earlier work of Killip–Visan–Zhang in [13] for cubic NLS in $2d$ which is mass critical. In their subsequent work, they considered the cubic NLS in $1d$ and proposed an alternative and simpler method to prove the weak well-posedness. Their proof relies on a compactness property which holds true only for the subcritical problem.

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