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Condition spectra of special operators and condition spectra preservers [☆]

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A B S T R A C T

Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. Descriptions of the condition spectra of various special classes of operators are established. As application, characterization is obtained for maps on $\mathcal{L}(\mathcal{H})$ leaving invariant the condition spectrum or the condition spectral radius of the product of operators. Analogous results are obtained for Jordan triple product of operators.

1. Introduction

Throughout, $\mathcal{H}$ will denote a Hilbert space over the complex field $\mathbb{C}$ and $\mathcal{L}(\mathcal{H})$ will denote the algebra of all bounded linear operators on $\mathcal{H}$ with unit I. For $T \in \mathcal{L}(\mathcal{H})$ we write $\sigma(T)$ for its spectrum and $\|T\|$ the (spectral) norm of T. For $0 < \varepsilon < 1$, the ε-condition spectrum of T, $\sigma_\varepsilon(T)$, is defined by

$$\sigma_\varepsilon(T) := \{z \in \mathbb{C} : \|z - T\|\|(z - T)^{-1}\| \geq \varepsilon^{-1}\}$$

with the convention that $\|z - T\|\|(z - T)^{-1}\| = \infty$ when $z - T$ is not invertible, and is a nonempty set, compact, perfect set (no isolated points) and always contains the usual spectrum $\sigma(T)$ of T; see [10]. Unlike the spectrum, which is a purely algebraic concept, the ε-condition spectrum is a geometric notion depending on the norm. The ε-condition spectral radius of T, $r_\varepsilon(T)$, is given by

$$r_\varepsilon(T) := \sup\{|z| : z \in \sigma_\varepsilon(T)\}.$$

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Condition spectra are a useful tool in numerical computations, and arise naturally in solving operator equations. They measure the computational stability of solving a linear system, especially in those involving the measure of the sensitivity of the answer to a problem to small changes in the initial data of the problem. For more details and basic facts concerning condition spectrum and its connection with the pseudo spectrum we refer the reader to [10] and the monograph [12].

In this paper, we compute in the next section the ε-condition spectrum and the ε-condition spectral radius of some special classes of operators. For example, we show that one can determine the condition spectra of self-adjoint operators, orthogonal projections and rank one nilpotent operators. As application, we characterize in Sections 3–4 maps ϕ on $\mathcal{L}(\mathcal{H})$ that preserve the condition spectrum or the condition spectral radius of product of operators or of Jordan triple product of operators, that is, those maps ϕ for which

$$\delta_\varepsilon(\phi(T) \circ \phi(S)) = \delta_\varepsilon(T \circ S) \qquad (T, S \in \mathcal{L}(\mathcal{H})),$$

where $\delta_\varepsilon(.)$ stands for $\sigma_\varepsilon(.)$ or $r_\varepsilon(.)$ and $T \circ S = TS$ or TST. It is shown that such a map is always a unitary similarity transform multiplied by a unit circle-valued functional. We note that the study of linear preservers of condition spectrum and pseudo spectrum was done by Kumar and Kulkarni [11], and nonlinear preservers of pseudo spectra have also been studied in some recent papers [1,2,4,6,7].

2. Some properties of the condition spectra

In this section, we give some properties of the condition spectra and we compute the condition spectra of some special classes of operators. First, we fix some notation. The inner product on $\mathcal{H}$ will be denoted by $\langle ., . \rangle$. For $x, f \in \mathcal{H}$, as usual we denote by $x \otimes f$ the rank at most one operator on $\mathcal{H}$ given by $z \mapsto \langle z, f\rangle x$, and all at most rank one operators in $\mathcal{L}(\mathcal{H})$ can be written into this form. For an operator $T \in \mathcal{L}(\mathcal{H})$ we will denote by $r(T)$, T^*, and T^{tr}, the spectral radius of T, the adjoint of T, and the transpose of T relative to an arbitrary but fixed orthogonal basis of $\mathcal{H}$, respectively. For a subset σ of $\mathbb{C}$ we will denote by $\overline{\sigma}$ the complex conjugation set of σ. It is straightforward that $\sigma_\varepsilon(T^{tr}) = \sigma_\varepsilon(T)$ and $\sigma_\varepsilon(T^*) = \overline{\sigma_\varepsilon(T)}$.

The following properties are useful; see [10].

Proposition 2.1. *For $0 < \varepsilon < 1$ and $T \in \mathcal{L}(\mathcal{H})$, the following statements hold.*

(i) *$\sigma_\varepsilon(T) = \sigma(T)$ if and only if T is a scalar multiple of the identity.*
(ii) *$\sigma_\varepsilon(\alpha + \beta T) = \alpha + \beta\sigma_\varepsilon(T)$ for all $\alpha, \beta \in \mathbb{C}$.*
(iii) *For every unitary operator $U \in \mathcal{L}(\mathcal{H})$, $\sigma_\varepsilon(UTU^*) = \sigma_\varepsilon(T)$.*

In the sequel, for $r \geq 0$ and $a \in \mathbb{C}$ we will denote by $\overline{D}(a, r)$ the closed disc of $\mathbb{C}$ centered at a and of radius r. The following result describes the condition spectrum of self-adjoint operators in terms of their usual spectrums.

Proposition 2.2. *Let $0 < \varepsilon < 1$ and $T \in \mathcal{L}(\mathcal{H})$ be a self-adjoint operator. Then*

$$\sigma_\varepsilon(T) = \bigcup_{\alpha,\beta\in\sigma(T)} \overline{D}\left(\frac{\alpha - \beta\varepsilon^2}{1-\varepsilon^2}, \frac{\varepsilon|\alpha - \beta|}{1-\varepsilon^2}\right).$$

Proof. Note that, for every $z \in \mathbb{C}$, the operator $z - T$ is normal. Thus, for every $z \in \sigma_\varepsilon(T) \setminus \sigma(T)$, we have

$$\begin{aligned}\|z - T\|\|(z - T)^{-1}\| \geq \varepsilon^{-1} &\Leftrightarrow r(z - T)r((z - T)^{-1}) \geq \varepsilon^{-1}\\ &\Leftrightarrow d(z, \sigma(T)) \leq \varepsilon r(z - T),\end{aligned}$$