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Limit cycles of discontinuous piecewise polynomial vector fields

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A R T I C L E I N F O

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A B S T R A C T

When the first average function is non-zero we provide an upper bound for the maximum number of limit cycles bifurcating from the periodic solutions of the center $\dot{x} = -y((x^2 + y^2)/2)^m$ and $\dot{y} = x((x^2 + y^2)/2)^m$ with $m \geq 1$, when we perturb it inside a class of discontinuous piecewise polynomial vector fields of degree n with k pieces. The positive integers m, n and k are arbitrary. The main tool used for proving our results is the averaging theory for discontinuous piecewise vector fields.

1. Introduction and statement of the main result

One of the main problems inside the qualitative theory of real planar differential systems is the determination of their limit cycles. The notion of a *limit cycle* of a planar differential system was defined by Poincaré [27] as a periodic orbit isolated in the set of all periodic orbits of the differential system. Van der Pol [28], Liénard [20] and Andronov [1] at the end of 1920s proved that a periodic orbit of a self-sustained oscillation occurring in a vacuum tube circuit was a limit cycle in the sense defined by Poincaré. After these results on the existence, non-existence and other properties of the limit cycles, these were studied with interest by mathematicians and physicists, and more recently also by many scientists of different areas (see for instance the books [10,32]).

In the last part of the XIX century Poincaré [27] defined the notion of a *center* of a real planar differential system, i.e. of an isolated equilibrium point having a neighborhood such that all the orbits of this neighborhood are periodic with the unique exception of the equilibrium point. Later on one way to produce limit cycles is by perturbing the periodic orbits of a center [29].

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Iliev [19] in 1999 considered the polynomial vector fields

$$\mathcal{X}(x,y) = \big(-y + \varepsilon P(x,y,\varepsilon), x + \varepsilon Q(x,y,\varepsilon) \big),$$

of degree $n > 1$ (i.e. the maximum of the degrees of polynomials P and Q is n), which depend analytically on the small parameter ε, and he studied how many limit cycles can bifurcate from the periodic orbits of the linear center $\dot{x} = -y$, $\dot{y} = x$ when $\varepsilon > 0$ is sufficiently small.

Buică, Giné and Llibre [8] in 2010 studied the same problem of Iliev but for the polynomial vector fields

$$\mathcal{X}(x,y) = \left(-y \left(\frac{x^2+y^2}{2} \right)^m + \varepsilon P(x,y,\varepsilon), x \left(\frac{x^2+y^2}{2} \right)^m + \varepsilon Q(x,y,\varepsilon) \right),$$

of degree the maximum of $2m+1$ and n being the maximum of the degrees of the polynomials P and Q, where again ε is a small parameter, and $m \geq 1$ is an integer. Of course, now the limit cycles bifurcate from the periodic solutions of the nonlinear center $\dot{x} = -y((x^2+y^2)/2)^m$, $\dot{y} = x((x^2+y^2)/2)^m$.

Andronov, Vitt and Khaikin [2] started the study of the continuous and discontinuous piecewise differential systems. These systems play an important role inside the nonlinear dynamical systems. They appeared in a natural way in nonlinear engineering models, and later on in electronic engineering, nonlinear control systems, biology, etc.; see for instance the books of di Bernardo, Budd, Champneys and Kowalczyk [5], Simpson [31], and the survey of Makarenkov and Lamb [26], and the hundreds of references quoted in these last three works.

There are many studies of the limit cycles of continuous and discontinuous piecewise differential systems in $\mathbb{R}^2$ with two pieces separated by a straight line. In general these differential systems are linear, see for instance [3,6,9,11–18,22–24,30]. But there are very few works of continuous and discontinuous piecewise differential systems with an arbitrary number k of pieces.

The objective of this paper is to study the number of limit cycles which can bifurcate from the center $\dot{x} = -y((x^2+y^2)/2)^m$, $\dot{y} = x((x^2+y^2)/2)^m$, when it is perturbed inside a class of discontinuous piecewise polynomial differential systems of degree n with k pieces.

More precisely, we consider the polynomial planar vector field

$$\mathcal{X} = \mathcal{X}(x,y) = \left(-y \left(\frac{x^2+y^2}{2} \right)^m, x \left(\frac{x^2+y^2}{2} \right)^m \right),$$

with either $m = 0$ (linear center) or m a positive integer (nonlinear center), and we perturb $\mathcal{X}$ with a discontinuous piecewise polynomial vector field as follows

$$\mathcal{X}_\varepsilon = \mathcal{X}_\varepsilon(x,y) = \mathcal{X}(x,y) + \varepsilon \sum_{i=1}^{k} \chi_{S_i}(x,y) \big(P_i(x,y), Q_i(x,y) \big),$$

where P_i and Q_i are polynomials of degree at most n, the characteristic function χ_K of a set $K \subset \mathbb{R}^2$ is defined by

$$\chi_K(x,y) = \begin{cases} 1 & \text{if } (x,y) \in K, \\ 0 & \text{if } (x,y) \notin K, \end{cases}$$

and the sets $S_1, \ldots, S_k$ satisfying $\cup_{i=1}^k \overline{S_i} = \mathbb{R}^2$ and $S_i \cap S_j = \emptyset$ for $i \neq j$ are defined as follows. For a given positive integer k consider k angles $0 \leq \theta_1 < \ldots < \theta_k < 2\pi$. Then the discontinuity set Σ for the discontinuous piecewise polynomial differential vector field $\mathcal{X}_\varepsilon$ is $\Sigma = \cup_{i=1}^k L_i$, where L_i is the ray starting at the origin and passing through the point $(\cos\theta_i, \sin\theta_i)$ for $i = 1, \ldots, k$, S_i is the interior of the sector with