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Liouville theorems for supersolutions of semilinear elliptic equations with drift terms in exterior domains

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A B S T R A C T

In this paper, we prove nonexistence of positive supersolutions of a semilinear equation $-\mathrm{div}\,(A(x)\nabla u) + \mathbf{b}(x) \cdot \nabla u = f(u)$ in exterior domains in $\mathbb{R}^n$ $(n \geq 3)$, where $A(x)$ is bounded and uniformly elliptic, $\mathbf{b}(x) = O(|x|^{-1})$, $\mathrm{div}\,\mathbf{b} = 0$ and f is a continuous and positive function in $(0, \infty)$ satisfying $f(u) \sim u^q$ as $u \to 0$ with $q \leq n/(n-2)$. Furthermore, we investigate general conditions on $\mathbf{b}$ and f for nonexistence of positive supersolutions.

1. Introduction

We consider nonexistence of positive (weak) solutions to differential inequalities

$$\mathcal{L}u := -\mathrm{div}\,(A(x)\nabla u) + \mathbf{b}(x) \cdot \nabla u \geq f(x, u) \quad \text{in } \mathbb{R}^n \setminus \overline{B_R}, \tag{1}$$

where $n \geq 3$ and B_R is a ball of radius $R > 0$ centered at the origin and $A = A(x)$ is a bounded measurable matrix-valued function which satisfies

$$\|A\|_{L^\infty(\mathbb{R}^n)} < \infty, \quad (A(x)\xi) \cdot \xi \geq |\xi|, \quad \forall x \in \mathbb{R}^n,\ \xi \in \mathbb{R}^n.$$

Throughout the paper, we also assume that the vector-valued function $\mathbf{b} = \mathbf{b}(x)$ belongs to $(L^2_{\mathrm{loc}}(\mathbb{R}^n))^n$ and $f : \mathbb{R}^n \times (0, \infty) \to (0, \infty)$ is a continuous function. Specific conditions on $\mathbf{b}$ and f will be described later.

Gidas [6] and Gidas and Spruck [7] proved the nonexistence of positive C^2 supersolutions of

$$-\triangle u + \frac{\beta x}{|x|^2} \cdot \nabla u = u^q \quad \text{in } \mathbb{R}^n \setminus \overline{B_R} \tag{2}$$

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for $1 < q \le (n-\beta)/(n-\beta-2)$. When $\beta = 0$, the range of the exponent is

$$1 < q \le \frac{n}{(n-2)}. \tag{3}$$

Note that if $\beta < 0$, then $1 < (n-\beta)/(n-\beta-2) < n/(n-2)$. Therefore, when $\mathbf{b} = O(|x|^{-1})$ and $\mathbf{b} \neq 0$, the nonexistence range (3) changes, in general. The exponent $(n-\beta)/(n-\beta-2)$ is sharp. Indeed, for $-\infty < \beta < n-2$ and $q > (n-\beta)/(n-\beta-2)$, the equation (2) has a positive solution $u(x) = c(n,q)|x|^{-2/(q-1)}$.

This type of nonexistence theorem has been extended for more general supersolutions of linear and nonlinear equations by many authors, see e.g. [22,19,17,18,5,12–14,4,1,2].

Recently, Armstrong and Sirakov [4] treated a wide class of second order (nonlinear) elliptic differential operators and nonlinearities. They developed a new method to show nonexistence of supersolutions of the equation

$$-Q[u] = f(x,u) \quad \text{in } \mathbb{R}^n \setminus \overline{B_R}, \tag{4}$$

where $Q[u]$ is several homogeneous elliptic differential operators with general nonlinearity $f(x,u)$. In particular, they proved that if $Q[u] = \triangle u$ and if $f(x,u) = f(u)$ satisfying $\liminf_{s\to 0} s^{-n/(n-2)} f(s) > 0$, then the equation (4) has no positive supersolutions.

On the other hand, Kondratiev et al. [14] gave a sufficient condition on $\mathbf{b}$ to assure the nonexistence of positive supersolutions of (1) for $f(x,u) = u^q$ with $1 < q \le n/(n-2)$. In [14], it was assumed that $A(x)$ is Hölder continuous and periodic with the same period, $\mathbf{b}$ satisfies some Kato type conditions, moreover,

$$|||\mathbf{b}||| = \left\{ C > 0;\ \frac{\int_{\mathbb{R}^n} |\mathbf{b}|^2 \phi^2\, dx}{\int_{\mathbb{R}^n} |\nabla\phi|^2\, dx} \le C^2 \quad \forall \phi \in C_c^\infty(\mathbb{R}^n) \right\} \tag{5}$$

is sufficiently small in some sense. Under these conditions, it was proved that (1) has no positive weak supersolutions if and only if $q \le n/(n-2)$.

In this paper, we give new sufficient conditions on $\mathbf{b}$ and f for nonexistence of positive supersolutions, using methods in [4] and techniques of (nonlinear) potential theory (see e.g. [11,21,15,10]). We shall prove the following:

Theorem 1. *Suppose that vector field* $\mathbf{b} = \mathbf{b}_0 + \mathbf{b}_1$ *satisfies*

$$|||\mathbf{b}_0||| < \infty \quad \textit{and} \quad \operatorname{div} \mathbf{b}_0 = 0 \quad \textit{in } \mathbb{R}^n \tag{6}$$

and

$$\mathbf{b}_1 \in (L^{n,1}(\mathbb{R}^n \setminus \overline{B_R}))^n \quad \textit{for some } R \ge 0. \tag{7}$$

Assume that $f(x,u) = |x|^{-\gamma} g(u)$, $\gamma < 2$ *and*

$$\liminf_{s\to 0} s^{-q} g(s) > 0$$

for $q = 1 + (2-\gamma)/(n-2)$. *Then* (1) *has no positive weak solutions.*

Here, $L^{p,\sigma}(\Omega)$ is a Lorentz space (see Section 2 for details). When $A(x) = I$, $\mathbf{b}_1 = 0$ and $f(x,u) = u^q$, Theorem 1 becomes as follows: