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Stochastic representations for the wave equation on graphs and their scaling limits

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A R T I C L E I N F O

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A B S T R A C T

This paper is devoted to an interacting particle system that provides probabilistic interpretation of the wave equation on graphs. A Feynman–Kac-type formula is established, connecting the expectation of the process with the wave equation on graphs. Non-asymptotic L^2 estimates are presented. It is then shown that the high-density hydrodynamic limit of the system is given by the wave equation in Euclidean space. The sharpness of scaling limit result is demonstrated by a phase transition phenomenon.

1. Introduction

Stochastic representations for solutions of elliptic and parabolic equations are well known since the earliest works by Kakutani [14], Kac [12], etc. They are constructed by Markov processes describing the random motion of a particle and thus also known as "stochastic solutions". These two types of PDEs have been studied intensively and thoroughly. Yet there has not been much progress on stochastic solutions of linear hyperbolic PDEs. A valuable survey is Hersh [10]. Goldstein [8] and Kac [13] were the first to construct stochastic solutions of the one-dimensional telegrapher's equation under some special initial conditions, using "persistent random walk". More general results are derived later, see [15,9,19,7,11]. Since most of those constructions require the solution of the wave equation associated to the telegrapher's equation, they cannot deal with the wave equation itself.

Only recently are there advances in stochastic solutions of the wave equation. Dalang, Mueller and Tribe [5] used the formulae of solutions to wave equations to construct stochastic solutions in one to three dimensional Euclidean spaces. Bakhtin and Mueller [1] defined "stochastic cascades" to solve one-dimensional semilinear wave equation. Pal and Shkolnikov [20] defined "intertwined diffusion processes" and showed their connections to the hyperbolic PDEs. Yet there is no explicit representation for solutions, and it cannot deal

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with initial value problems. Chatterjee [4] derived a family of functions in bounded domains that satisfy the wave equation, using Brownian motion and a Cauchy random variable. This is the first result about bounded domains, although it is still unable to handle prescribed initial and boundary data. Plyukhin [21] analyzed the inability of single-particle motion, and defined a stochastic process describing the movements and transitions of a large number particles moving along positive and negative directions of the Cartesian axes, to use their distributions to solve equations including the wave equation. There is no rigorous analysis and initial–boundary value problems were not addressed, either. Probabilistic interpretations of the wave equation still need exploration.

On the other hand, interacting particle systems have been successfully used as models for many differential equations. Kurtz [17,18] considered Markov population processes with finite types of individual, and established law of large numbers approximation and diffusion approximation of systems of finitely many ODEs. Then Kotelenez [16], Blount [3] and many others extended the results to parabolic PDEs by hydrodynamic limits. In parallel with those works, stochastic processes describing evolutions with infinite types of individual are also studied, see Eibeck and Wagner [6] and Barbour [2]. They are related to PDEs or systems of infinitely many ODEs.

In this paper we start from the wave equation on graphs, which is a system of finitely or infinitely many second-order linear ODEs. It is an approximation of the wave equation in Euclidean spaces and arises from many physics and engineering studies including spring networks, LC circuits, etc. An interacting particle system is constructed as the probabilistic model for this. There are key features distinguishing it from most existing models for other equations. One is that the particles are located not only on the nodes, but also on the edges of the graph. This follows from physics interpretations of the problem. Besides, we have dimension-free estimates for the system, which does not depend on the number of vertices in the graph. Hence infinite graphs such as $\mathbb{Z}^d$ are easily analyzed. What is more, a phase transition phenomenon demonstrates the sharpness of the scaling result.

We use quadruple $G = (V, E, K, m)$ to denote a graph to be discussed throughout the paper, where V and E are sets of nodes and edges, respectively; $K = (k_{xy})_{x,y\in V}$ is the weight matrix, $k_{xy} = k_{yx} \geq 0$, and $k_{xy} = 0$ if there is no edge between x and y; $m = (m_x)_{x\in V}$ is a function on V taking values in $\mathbb{R}_+$. We further suppose the graph is embedded in some Hilbert space $\mathcal{H}$, i.e. the nodes are elements in that space, where the inner product and norm are represented by "$\cdot$" and "$|\cdot|$", respectively. In this paper, V is either finite or countable, and the edges are undirected. There can be at most one edge between any pair of nodes, and there is no self-edges.

Definition 1. The Dirichlet initial–boundary value problem (IBVP) of the wave equation on G is

$$\begin{cases} m_x \dfrac{\mathrm{d}^2 u}{\mathrm{d}t^2}(x,t) = \displaystyle\sum_{y\in V} k_{xy}[u(y,t) - u(x,t)], \ (x,t) \in V_0 \times [0, +\infty), \\ u(x,0) = \varphi(x), \ \dfrac{\mathrm{d}u}{\mathrm{d}t}(x,0) = \psi(x), \ x \in V; \\ u(x,t) = \varphi(x), \ (x,t) \in V_1 \times [0, +\infty). \end{cases} \tag{1.1}$$

Here V_0 and V_1 are two disjoint subsets of V and $V_0 \cup V_1 = V$. φ and ψ are real-valued functions on V, and $\psi|_{V_1} = 0$.

Our process is rather natural and easy to analyze. Thanks to linearity of this problem, the expectation of the process is directly linked to the wave equation through a Feynman–Kac-type formula. Under some regularity conditions, we can define an interacting particle system $\{f_t : t \geq 0\}$ whose states are functions on $V \cup E$. The initial state is determined by the initial and boundary data in (1.1). Then Theorem 3.1 states that

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