J. Math. Anal. Appl. • • • (• • • •) • • •–• • •

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Wave propagation in an infectious disease model ☆

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A R T I C L E I N F O

Article history:
Received 15 December 2015
Available online xxxx
Submitted by Y. Du

Keywords:
Infectious disease model
Lyapunov functional
Schauder fixed point theorem
Traveling wave
Upper and lower solutions

A B S T R A C T

This paper is devoted to the study of the wave propagation in a reaction-convection infectious disease model with a spatio-temporal delay. Previous numerical studies have demonstrated the existence of traveling wave fronts for the system and obtained a critical value c^*, which is the minimal wave speed of the traveling waves. In the present paper, we provide a complete and rigorous proof. To overcome the difficulty due to the lack of monotonicity for the system, we construct a pair of upper and lower solutions, and then apply the Schauder fixed point theorem to establish the existence of a nonnegative solution for the wave equation on a bounded interval. Moreover, we use a limiting argument and in turn generate the solution on the unbounded interval $\mathbb{R}$. In particular, by constructing a suitable Lyapunov functional, we further show that the traveling wave solution converges to the epidemic equilibrium point as $t = +\infty$.

1. Introduction

In 2009, Li and Zou [8] used an SIR model to derive the following reaction-convection infectious disease model with a spatio-temporal delay

$$\begin{cases} S_t(t,x) = D_S S_{xx}(t,x) + \mu - dS(t,x) - rI(t,x)S(t,x), \\ I_t(t,x) = D_I I_{xx}(t,x) - \beta I(t,x) + \epsilon r \int_{-\infty}^{+\infty} f_\alpha(x-y) I(t-\tau,y) S(t-\tau,y)dy, \end{cases} \tag{1.1}$$

where S and I represent the densities of the susceptible and infective individuals at time t and position $x \in \mathbb{R}$, respectively, D_S and D_I are the corresponding diffusion rates. $\mu > 0$ is a constant recruiting rate, d is the natural death rate, $r > 0$ denotes the infection rate, ϵ measures the proportion of infected individuals that can survive the latent period, and the delay τ represents the latency length of the infective

☆ Research partially supported by the NSF of Guangdong Province (2016A030313426) and the HLUCF of South China Normal University (2016YN30).

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http://dx.doi.org/10.1016/j.jmaa.2016.11.051

disease. $\beta = \sigma + \gamma + d$, σ, and γ are the disease-induced mortality rate and the recovery rate, respectively, $f_\alpha(x) = \frac{1}{\sqrt{4\pi\alpha}} e^{-\frac{x^2}{4\alpha}}$. Readers may refer to [8] for a precise interpretation of the biological implications for system (1.1).

Based on an abstract treatment, Li and Zou [8] addressed the existence, uniqueness and positivity of solutions of system (1.1). In addition, in view of the numeric simulations, they explored the existence of traveling wave fronts for system (1.1), and obtained a critical value c^*, which is the minimal wave speed c of the traveling wave fronts, i.e., system (1.1) admits traveling waves with wave speed $c \geq c^*$ but no such traveling waves with wave speed $c < c^*$. In this paper, we provide the first rigorous mathematical proof of the existence of traveling wave solutions for system (1.1).

Since system (1.1) does not satisfy the comparison principle and possess monotone properties, it is difficult to apply the general theory regarding the existence of traveling wave solutions for monotone system developed by Huang and Zou [7], Liang and Zhao [10], Ma [11], Wang, Li and Ruan [14], Wu and Zou [15], and the references cited therein. In this paper, motivated by previous works [1–4,9,13,16], we use an iteration process [1] (see also [2–4,9,13,16]) to construct a pair of upper and lower solutions $(\overline{S}, \overline{I})$ and $(\underline{S}, \underline{I})$. Using the constructed pair of upper and lower solutions, we build an appropriately invariant cone Γ_X of initial functions defined on a bounded interval, and we then apply the Schauder fixed point theorem for this cone to establish the existence of a nonnegative solution of (2.2) on the bounded interval, which serves as a candidate for the traveling wave solution for (2.2) on the unbounded interval $\mathbb{R}$. Furthermore, following the idea proposed in [17] (also see [2–4,13]), we employ a limiting argument to generate the solution on $\mathbb{R}$.

We should stress that according to the construction of the upper and lower solutions, the obtained traveling wave (S, I) is a nonnegative solution for (2.2) on $\mathbb{R}$ with $(S, I)(-\infty) = (1, 0)$. To demonstrate the existence of a traveling wave connecting the disease-free and endemic equilibrium, we need to prove that $(S, I)(+\infty) = (S^*, I^*)$. It is well known that the method of Lyapunov functionals [5] is a direct and effective approach for studying the global stability of delayed differential systems. However, it is challenging and difficult to construct a suitable Lyapouov functional for the differential systems with delay. In the present paper, inspired by the ideas proposed by [2–4,9], we successfully construct a Lyapunov functional and then show that $(S, I)(+\infty) = (S^*, I^*)$. We also comment that the construction of the Lyapunov functional is nontrivial and difficult because the corresponding wave profile system (2.2) is a second order functional differential system of mixed type (i.e., with both advanced and non-local delayed arguments).

The remainder of this paper is organized as follows. In Section 2, we give an important lemma and state the main results. In Section 3, we derive the preliminary results, including the construction of the upper and lower solutions, and the existence of the solution to (2.2) on a bounded interval. The proof of Theorem 2.1 is given in Section 4. Finally, we provide a brief discussion.

2. Main results

In this section, we state the main results. For simplicity, let

$$\widetilde{S}(t,x) = \frac{d}{\mu} S(t, x\sqrt{D_I}), \quad \widetilde{I}(t,x) = \frac{d}{\mu} S(t, x\sqrt{D_I}),$$

and

$$\widetilde{d} = \frac{D_S}{D_I}, \quad \widetilde{r} = \frac{r\mu}{d}, \quad k = \frac{\epsilon r \mu}{d}.$$

By dropping the tilde for convenience, we then consider the following system

$$\begin{cases} S_t(t,x) = dS_{xx}(t,x) + \mu(1 - S(t,x)) - rI(t,x)S(t,x), \\ I_t(t,x) = I_{xx}(t,x) - \beta I(t,x) + k \int\limits_{-\infty}^{+\infty} f_\alpha(x-y) I(t-\tau, y) S(t-\tau, y) dy. \end{cases} \tag{2.1}$$

Please cite this article in press as: Z. Xu, Wave propagation in an infectious disease model, J. Math. Anal. Appl. (2017), http://dx.doi.org/10.1016/j.jmaa.2016.11.051