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Eigenvalue inequalities and absence of threshold resonances for waveguide junctions

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ABSTRACT

Let $\Lambda \subset \mathbb{R}^d$ be a domain consisting of several cylinders attached to a bounded center. One says that Λ admits a threshold resonance if there exists a non-trivial bounded function u solving $-\Delta u = \nu u$ in Λ and vanishing at the boundary, where ν is the bottom of the essential spectrum of the Dirichlet Laplacian in Λ. We give a sufficient condition for the absence of threshold resonances in terms of the Laplacian eigenvalues on the center. The proof is elementary and is based on the min–max principle. Some two- and three-dimensional examples and applications to the study of Laplacians on thin networks are discussed.

1. Introduction

Let $\Lambda \subset \mathbb{R}^d$, $d \geq 2$, be a connected Lipschitz domain which can be represented as a family of several half-infinite cylinders attached to a bounded domain. More precisely, we assume that there exist bounded connected Lipschitz domains $\omega_j \subset \mathbb{R}^{d-1}$, called cross-sections, and n non-intersecting half-infinite cylinders $B_1, \dots, B_n \subset \Lambda$, isometric respectively to $\mathbb{R}_+ \times \omega_j$, $\mathbb{R}_+ := (0, +\infty)$, such that Λ coincides with the union $B_1 \cup \ldots \cup B_n$ outside a compact set, see Fig. 1(a). The cylinders B_j will be called *branches*, the connected bounded domain $C := \Lambda \setminus \overline{B_1 \cup \ldots \cup B_n}$ will be called *center*, and we assume that the boundary of C is Lipschitz too. We call such a domain Λ a *star waveguide*. Remark that the choice of a center in not unique: any center can be enlarged by including finite pieces of the branches, see Fig. 1(b).

In the present work, we would like to establish some elementary conditions guaranteeing the non-existence of non-trivial bounded solutions to

$$-\Delta u = \nu u \text{ in } \Lambda\,, \quad u = 0 \text{ at } \partial\Lambda, \tag{1}$$

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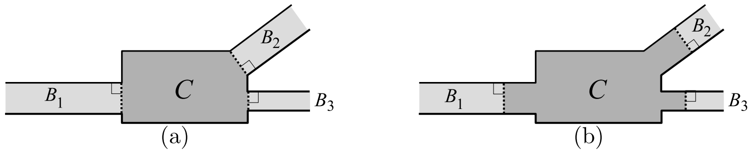

Fig. 1. (a) An example of a star waveguide Λ with three branches and a dark-shaded center. (b) An alternative choice of a center.

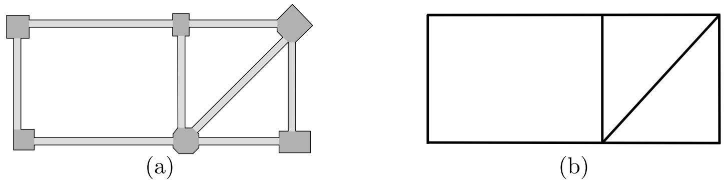

Fig. 2. (a) An example of a domain Ω_ε. The vertex parts are dark-shaded. (b) The associated one-dimensional skeleton X.

where ν is the bottom of the essential spectrum of the Dirichlet Laplacian $-\Delta_D^\Lambda$ acting in $L^2(\Lambda)$. It is standard to see that $\nu = \min \nu_j$, where ν_j is the lowest Dirichlet eigenvalue of the cross-section ω_j, and the spectrum of $-\Delta_D^\Lambda$ consists of the semi-axis $[\nu, \infty)$ and of a finite family of discrete eigenvalues $\lambda_j(-\Delta_D^\Lambda)$, $j \in \{1, \dots, N(\Lambda)\}$, while the case $N(\Lambda) = 0$ (no discrete eigenvalues) is possible. As shown e.g. in [19, Theorem 4], a non-trivial bounded solution of (1) exists iff the resolvent $z \mapsto (-\Delta_D^\Lambda - z)^{-1}$ has a pole at $z = \nu$, and in that case we say that Λ admits a threshold resonance.

The study of threshold resonances is motivated, in particular, by the analysis of Dirichlet Laplacians in systems of thin tubes collapsing onto a graph. Namely, for a small $\varepsilon > 0$, consider a domain $\Omega_\varepsilon \subset \mathbb{R}^d$ composed of finitely many cylinders ("edges") $B_{j,\varepsilon}$ isometric to $I_j \times (\varepsilon\omega)$ with $I_j := (0, \ell_j)$, $\ell_j \in \mathbb{R}_+$, $j \in \{1, \dots, J\}$, connected to a "network" through some bounded Lipschitz domains ("vertices") $C_{k,\varepsilon}$, see Fig. 2(a). (The case of nonidentical cross-sections is also possible but the formulations become more complicated.) We assume that the vertices $C_{k,\varepsilon}$ are isometric to εC_k, where the domains C_k are ε-independent, $k \in \{1, \dots, K\}$, and that the pieces are glued together in such a way that if one considers a vertex $C_{k,\varepsilon}$ and extends the attached edges to infinity, then one obtains a domain isometric to $\varepsilon\Lambda_k$, where Λ_k is an ε-independent star waveguide whose center is C_k.

Denote by $-\Delta_D^{\Omega_\varepsilon}$ the Dirichlet Laplacian in Ω_ε. In various applications, one is interested in the asymptotics of its eigenvalues $\lambda_m(-\Delta_D^{\Omega_\varepsilon})$ as ε tends to 0, see e.g. the monographs [11,25] and the reviews [15,17]. As the domain Ω_ε collapses onto it one-dimensional skeleton X composed from the intervals I_j coupled at the vertices, see Fig. 2(b), one may expect that the eigenvalue asymptotics might be determined by some effective operator acting on the functions defined on X. The results obtained by several authors, see e.g. [14,19], can be informally summarized as follows. Consider the star waveguides Λ_k associated to each vertex as described above, the associated Dirichlet Laplacians $-\Delta_D^{\Lambda_k}$ and their discrete eigenvalues $\lambda_j(-\Delta_D^\Lambda)$, $j \in \{1, \dots, N(\Lambda_k)\}$, $k \in \{1, \dots, K\}$, then the bottom of the essential spectrum is exactly the first Dirichlet eigenvalue ν of the cross-section ω, and the following holds as ε tends to 0: there exists $N \geq N(\Lambda_1) + \dots + N(\Lambda_K)$ such that

- for $m \in \{1, \dots, N\}$ there holds

$$\lambda_m(-\Delta_D^{\Omega_\varepsilon}) = \frac{a_m}{\varepsilon^2} + \mathcal{O}(e^{-c_m/\varepsilon}) \text{ with } a_m \in (0, \nu] \text{ and } c_m > 0,$$

- for any $m \geq 1$ there holds

$$\lambda_{N+m}(-\Delta_D^{\Omega_\varepsilon}) = \frac{\nu}{\varepsilon^2} + \mu_m + \mathcal{O}(\varepsilon),$$

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