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Regularity properties of non-negative sparsity sets

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A R T I C L E I N F O

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A B S T R A C T

This paper investigates regularity properties of two non-negative sparsity sets: non-negative sparse vectors, and low-rank positive semi-definite matrices. Novel formulae for their Mordukhovich normal cones are given and used to formulate sufficient conditions for non-convex notions of regularity to hold. Our results provide a useful tool for justifying the application of projection methods to certain rank constrained feasibility problems.

1. Introduction

The solutions of many optimization and reconstruction problems admit characterizations in terms of certain sparse objects. For example, it is sometimes possible to uniquely solve under-determined linear systems under addition assumptions of sparsity [12]. The difficulty arising in such formulations is in dealing with poorly behaved *sparsity functionals*. Two important examples of such functionals are the ℓ_0-"norm" for vectors, and the rank function for matrices. It is well known that sparsity functionals lead to problems involving non-convexity and NP-hard complexity (see, for example, [17,28]).

A popular approach to addressing the aforementioned difficulty is to employ *convex relaxations* [8,10,33], thus allowing for application of industrial strength non-linear solvers. For instance, the ℓ_1-norm promotes sparsity and has consequently been used as a surrogate for its ℓ_0 counterpart. Such relaxations come with varying strengths and theoretical guarantees. For an introduction to the topic, we refer the reader to [17, Ch. 4]. Whilst one may be able to exactly solve a relaxation, it is not always the case that this translates into a satisfactory sparse solution of the original problem.

An alternative approach involves attempting to deal with the original problem's non-convexity directly [1,6,20], and thus avoiding the potential complication of recovering a sparse solution from a convex relaxation. Here one can typically only give theoretical guarantees which apply locally (*i.e.,* within some

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neighborhood of a solution). In these cases, *regularity properties* of the constraint sets play an important role and can often be usefully formulated in the language of *normal cones*.

This paper investigates regularity properties of sparsity sets having additional non-negativity constraints. We focus on two such sets: non-negative sparse vectors, and low-rank positive semi-definite matrices. Simple, novel formulae for their Mordukhovich normal cones are given, and then used to formulate sufficient conditions to ensure various regularity properties hold. Implications for algorithms and applications are discussed, with particular attention given to *low-rank Euclidean distance matrix reconstruction*.

The remainder of this paper is organized as follows. In Section 2, we introduce notation and recall results which will be of use. In Section 3, we consider the non-negative sparse vector settings, before *lifting* results to their positive semi-definite counterparts. In Section 4, we deduce consequences of the results from the previous two sections including regularity properties for problems having non-negative sparsity sets. Finally, in Section 5, various example applications of problems in which non-negative sparsity sets arise are given.

2. Preliminaries and notation

Let $\mathbb{E}$ denote a finite dimensional real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$. Throughout this paper we focus on two such spaces. The first is $\mathbb{R}^m$ equipped with the standard inner product. The second is the set of real symmetric $m \times m$ matrices denoted $\mathbb{S}^m$ equipped with inner product

$$\langle X, Y\rangle := \operatorname{tr}\left(X^\top Y\right),$$

where $\operatorname{tr}(\cdot)$ (resp. $(\cdot)^\top$) denotes the trace (resp. transpose) of matrix. The induced norm is the *Frobenius norm* which is given by

$$\|X\| = \sqrt{\sum_{i=1}^m \sum_{j=1}^m X_{ij}^2}.$$

One may, of course, think of the Frobenius norm as treating matrices as “long vectors”.

The set of *positive (resp. negative) semi-definite* $m \times m$ matrices is denoted $\mathbb{S}^m_+$ (resp. $\mathbb{S}^m_-$) and we write $x \succeq 0$ (resp. $x \preceq 0$) to mean $x \in \mathbb{S}^m_+$ (resp. $x \in \mathbb{S}^m_-$). The set of $m \times m$ *orthogonal* (resp. *permutation*) matrices is denoted $\mathbb{O}^m$ (resp. $\mathbb{P}^m$).

The *projection* mapping onto the set $\Omega \subseteq \mathbb{E}$ is the set-valued mapping $P_\Omega : \mathbb{E} \rightrightarrows \Omega$ given by

$$P_\Omega(x) := \left\{ y \in \Omega : \|x - y\| \le \inf_{z\in\Omega} \|x - z\| \right\}.$$

When $P_\Omega(x) = \{y\}$ (*i.e.*, $P_\Omega(x)$ is a singleton) we write $P_\Omega(x) = y$.

In finite dimensions the *Mordukhovich normal cone* to the set $\Omega \subseteq \mathbb{E}$ at a point $\overline{x} \in \Omega$ can be represented as

$$N_\Omega(\overline{x}) = \{y \in \mathbb{E} : \exists (x_n), (y_n) \text{ s.t. } x_n \to \overline{x},\, y_n \to y,\, y_n \in \mathbb{R}_+(x_n - P_\Omega(x_n))\}.$$

For closed convex sets this simplifies to the classical *convex normal cone* given by

$$N^{\text{conv}}_\Omega(\overline{x}) := \{y \in \mathbb{E} : \langle y, x - \overline{x}\rangle \le 0,\, \forall x \in \Omega\},$$

but still remains useful in non-convex settings [27, Ch. 1]. The *proximal normal cone* to the set $\Omega \subseteq \mathbb{E}$ at a point $\overline{x} \in \Omega$ is given by