J. Math. Anal. Appl. 447 (2017) 988–1008

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

An accelerated exponential time integrator for semi-linear stochastic strongly damped wave equation with additive noise [☆]

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ARTICLE INFO

Article history:
Received 8 February 2016
Available online 11 October 2016
Submitted by A. Jentzen

Keywords:
Strongly damped wave equation
Infinite dimensional Wiener process
Spectral Galerkin method
Accelerated exponential time integrator
Strong approximation

ABSTRACT

This paper is concerned with the strong approximation of a semi-linear stochastic wave equation with strong damping, driven by additive noise. Based on a spatial discretization performed by a spectral Galerkin method, we introduce a kind of accelerated exponential time integrator involving linear functionals of the noise. Under appropriate assumptions, we provide error bounds for the proposed full-discrete scheme. It is shown that the scheme achieves higher strong order in time direction than the order of temporal regularity of the underlying problem, which allows for higher convergence rate than usual time-stepping schemes. Particularly for the space–time white noise case in two or three spatial dimensions, the scheme still exhibits a good convergence performance. Another striking finding is that, even for the velocity with low regularity, the scheme always promises first order strong convergence in time. Numerical examples are finally reported to confirm our theoretical findings.

1. Introduction

Great attention has been devoted in the last decade to numerical approximations of hyperbolic stochastic partial differential equations (SPDEs) (see, e.g., [1,2,4,6,10,12–14,24,28,30] and references therein). In the present work, we concentrate on a class of semi-linear SPDEs of second order with damping, described by

$$\begin{cases} u_{tt} = \alpha L u_t + Lu + F(u) + \dot{W}(t), & \text{in } \mathcal{D} \times (0,\ T], \\ u(\cdot,0) = u_0,\ u_t(\cdot,0) = v_0, & \text{in } \mathcal{D}, \\ u = 0, & \text{on } \partial\mathcal{D} \times (0,\ T], \end{cases} \tag{1.1}$$

☆ R.Q. was supported by Research Fund for Northeastern University at Qinhuangdao (No. XNB201429), Fundamental Research Funds for Central Universities (No. N130323015), Science and Technology Research Founds for Colleges and Universities in Hebei Province (No. Z2014040), NSF of Hebei Province (No. A2015501102). X.W. was supported by NSF of China (No. 11671405, No. 11301550, No. 11571373, No. 11401594), NSF of Hunan Province (No. 2016JJ3137), Innovation Program of Central South University (No. 10900-506010101) and Program of Shenghua Yuying at Central South University.

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http://dx.doi.org/10.1016/j.jmaa.2016.09.052
0022-247X/

where $T \in (0, \infty)$, $\mathcal{D} \subset \mathbb{R}^d$, $d = 1, 2, 3$, is a bounded open domain with smooth boundary $\partial\mathcal{D}$, and where $L := \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\big(l_{ij}(x)\frac{\partial}{\partial x_j}\big), x \in \mathcal{D}$ is a linear second-order elliptic operator with smooth coefficients $\{l_{ij}\}_{i,j=1}^{d}$ being uniformly positive definite. Let $\alpha > 0$ be a fixed positive constant and let $\{W(t)\}_{t\in[0,T]}$ be a (possibly cylindrical) Q-Wiener process on $H := L_2(\mathcal{D})$, defined on a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\in[0,T]})$ with respect to the normal filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$. The initial data u_0, v_0 are assumed to be $\mathcal{F}_0$-measurable random variables.

The deterministic counterpart of (1.1), called strongly damped wave equation (SDWE), occurs in a wide range of applications such as modeling motion of viscoelastic materials [7,18,19]. From both the theoretical and numerical point of view, the deterministic problem has been extensively studied (e.g., [11,16,26]). However, the corresponding stochastic strongly damped wave equations are theoretically and numerically far from well-understood from existing literature [5,21]. In the classical monograph [5], a stochastic strongly damped wave equation with multiplicative noise was discussed and a unique mild solution was established. In our recent publication [21], we analyzed the regularity properties of the mild solution to (1.1) and examine error estimates of a full discretization, done by the finite element spatial approximation together with the well-known linear implicit Euler temporal discretization. It was shown there that, for a certain class of stochastic SDWEs, the convergence rates of the usual full discretization coincide with the space–time regularity properties of the mild solution (see Theorem 2.4 in [21]). In this article, we aim to introduce a so-called accelerated exponential time integrator for the problem (1.1), which, as we will show later, promises higher convergence order in time than the derived order of temporal regularity of the underlying problem.

Different from the usual Euler-type time-stepping schemes using the basic increments of the driving Wiener process [1,3,15,17,22,23,29,32], the accelerated exponential time integrators rely on suitable linear functionals of the Wiener process and usually attain higher approximation orders in time [8,9,30,31]. In 2009, such scheme was first constructed by Jentzen and Kloeden [8] for semi-linear parabolic SPDEs with additive space–time white noise. The order barrier in the numerical approximation of parabolic SPDEs was overcome and a strong convergence rate of order $1-\epsilon$ in time was obtained, for arbitrarily small $\epsilon > 0$, unfortunately with seriously restrictive commutativity condition imposed on the non-linear term (see [8, Assumption 2.4]). Afterwards, the accelerated schemes were extended to solve a larger class of parabolic SPDEs with more general noise and the error bounds were analyzed under relaxed conditions on the non-linear term [9,31]. Furthermore, the accelerated scheme was successfully adapted to solve semilinear stochastic wave equations and the order barrier $\frac{1}{2}$ was went beyond [30].

Following the idea of the acceleration technique, we discretize the considered problem (1.1) in space by a spectral Galerkin method and in time by an exponential integrator involving linear functionals of the noise. To analyze the resulting error bounds, we formulate mild assumptions on the nonlinear mapping F (see Assumption 2.3), to allow for a large class of nonlinear Nemytskii operators. Additionally, we assume the covariance operator $Q\colon L_2(\mathcal{D}) \to L_2(\mathcal{D})$ of the Wiener process obeys

$$\|A^{\frac{\gamma-1}{2}} Q^{\frac{1}{2}}\|_{\mathrm{HS}} < \infty \quad \text{for some } \gamma \in (-1, 2], \tag{1.2}$$

which covers both the space–time white noise case and the trace-class noise case. Here $A := -L$ with domain $D(A) = H^2(\mathcal{D}) \cap H_0^1(\mathcal{D})$. Under these assumptions, the continuous problem (1.1) possesses a unique mild solution with the displacement $u(t), t \in [0, T]$ taking values in $L^p(\Omega; \dot{H}^{1+\min\{\gamma,1\}})$ for $\gamma \in (-1, 2]$ and the velocity $v(t) := u_t(t), t \in [0, T]$ taking values in $L^p(\Omega; \dot{H}^{\gamma})$ for $\gamma \in [0, 2]$. Here $\dot{H}^s = D(A^{\frac{s}{2}}), s \in \mathbb{R}$ will be specified later. For $M, N \in \mathbb{N}$, let $(u_m^{N,M}, v_m^{N,M})'$ be the numerical approximations of $(u(t_m), v(t_m))'$ at the temporal grid points $t_m = mk$ for $m \in \{0, 1, ..., M\}$, produced by the proposed full discrete scheme with time-step $k = T/M$. Under certain assumptions, our convergence analysis shows, there exists a constant C such that, for all $N, M \in \mathbb{N}$,

$$\sup_{m\in\{0,1,\cdots,M\}} \|u(t_m) - u_m^{N,M}\|_{L^2(\Omega;\dot{H}^0)} \le C(k^{\min\{1+\gamma,1\}} + \lambda_{N+1}^{-\frac{1+\min\{\gamma,1\}}{2}}), \quad \gamma \in (-1, 2], \tag{1.3}$$