J. Math. Anal. Appl. 447 (2017) 1009–1026

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Completely hyperexpansive tuples of finite order

Sameer Chavan [a,*], V.M. Sholapurkar [b]

[a] *Indian Institute of Technology Kanpur, Kanpur-208016, India*
[b] *Center for Postgraduate Studies in Mathematics, S. P. College, Pune-411030, India*

ARTICLE INFO

Article history:
Received 24 January 2016
Available online 29 October 2016
Submitted by M. Mathieu

Keywords:
Completely monotone
Completely alternating
Joint subnormal
Completely hyperexpansive
Cauchy dual

ABSTRACT

We introduce and discuss a class of operator tuples, which we call completely hyperexpansive tuples of finite order. This class is in some sense antithetical to the class of completely hypercontractive tuples of finite order studied in the prequel of this paper. Motivated by Shimorin's notion of Cauchy dual operator, we also discuss a transform which sends certain completely hyperexpansive multishifts of finite order k to completely hypercontractive multishifts of respective order.

1. Preliminaries

The present work is a sequel to the paper [10]. In that paper, we discussed a class of functions, referred to as completely monotone functions of finite order, which in particular includes polynomials and completely monotone functions. The objective of the said paper was to study the operator-theoretic analogue of that class of functions. We observed that the new class of operator tuples, referred to as completely hypercontractive tuples of finite order, includes joint m-isometries and joint subnormal contractions. In some sense, that class of operator tuples provides a unified treatment of both these well known classes of operator tuples. However, among several examples of the completely hypercontractive tuples of finite order, there are examples of tuples which are neither m-isometries nor subnormals.

We recall the notation used in [10] for the ready reference. The symbol $\mathbb{N}$ stands for the set of non-negative integers; $\mathbb{N}$ forms a semigroup under addition. Let $\mathbb{N}^m$ denote the Cartesian product $\mathbb{N} \times \cdots \times \mathbb{N}$ (m times). Let $p \equiv (p_1, \cdots, p_m)$ and $n \equiv (n_1, \cdots, n_m)$ be in $\mathbb{N}^m$. We write $|p| := \sum_{i=1}^m p_i$ and $p \leq n$ if $p_i \leq n_i$ for $i = 1, \cdots, m$. For $n \in \mathbb{N}^m$, we let $n! := \prod_{i=1}^m n_i!$.

* Corresponding author.
E-mail addresses: chavan@iitk.ac.in (S. Chavan), vmshola@gmail.com (V.M. Sholapurkar).

http://dx.doi.org/10.1016/j.jmaa.2016.10.065
0022-247X/

For a real-valued map φ on $\mathbb{N}$, we define (backward and forward) difference operators ∇ and Δ as follows: $(\nabla\varphi)(n) = \varphi(n) - \varphi(n+1)$ and $(\Delta\varphi)(n) = \varphi(n+1) - \varphi(n)$. The operators ∇^n and Δ^n are inductively defined for all $n \in \mathbb{N}$ through the relations

$$\nabla^0\varphi = \Delta^0\varphi = \varphi,\ \nabla^n\varphi = \nabla(\nabla^{n-1}\varphi)(n \geq 1),\ \Delta^n\varphi = \Delta(\Delta^{n-1}\varphi)(n \geq 1).$$

A real-valued map φ on $\mathbb{N}$ is said to be *completely monotone* if $(\nabla^k\varphi)(n) \geq 0$ for all $n \geq 0$, $k \geq 0$. A real-valued map ψ on $\mathbb{N}$ is said to be *completely alternating* if $(\nabla^k\psi)(n) \leq 0$ for all $n \geq 0$, $k \geq 1$. Completely monotone maps on $\mathbb{N}$ form an extreme subset of the set of positive definite functions on $\mathbb{N}$, while completely alternating maps form an extreme subset of the set of negative definite functions on $\mathbb{N}$ (refer to [6]).

For a complex, infinite-dimensional, separable Hilbert space $\mathcal{H}$, let $B(\mathcal{H})$ denote the C^*-algebra of bounded linear operators on $\mathcal{H}$.

Completely hyperexpansive operators were introduced independently in [2] and [5]. An operator T in $B(\mathcal{H})$ is *completely hyperexpansive* if

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}T^{*k}T^k \leq 0 \text{ for all } n \geq 1. \tag{1.1}$$

It was observed in [5] that the condition (1.1) is equivalent to requiring, for every h in $\mathcal{H}$, the map $\psi_h(n) = \|T^n h\|^2$ to be completely alternating on $\mathbb{N}$. The symbiotic relationship between completely monotone and completely alternating maps carries over to subnormal contractions and completely hyperexpansive operators respectively, and this theme was focused upon in [5]. The reader is referred to [25] and [18] for the basic theory of hyperexpansive operators.

By a *commuting m-tuple* T on $\mathcal{H}$, we mean a tuple $(T_1, \cdots, T_m)$ of commuting bounded linear operators $T_1, \cdots, T_m$ on $\mathcal{H}$. For a commuting m-tuple T on $\mathcal{H}$, we interpret T^* to be $(T_1^*, \cdots, T_m^*)$, and T^p to be $T_1^{p_1}\cdots T_m^{p_m}$ for $p = (p_1, \cdots, p_m) \in \mathbb{N}^m$.

For the definitions and the basic theory of various spectra including the Taylor spectrum, the reader is referred to [14]. For a commuting m-tuple T, we reserve the symbols $\sigma(T), \sigma_l(T)$ and $\sigma_{ap}(T)$ for the Taylor spectrum, left-spectrum and approximate point spectrum of T respectively. The symbols $r(T)$ and $r_l(T)$ stand for the geometric spectral radii of $\sigma(T)$ and $\sigma_l(T)$ respectively. We recall that $r(T)$ and $r_l(T)$ are given by

$$r(T) := \sup\{\|z\|_2 : z \in \sigma(T)\},\ r_l(T) := \sup\{\|z\|_2 : z \in \sigma_l(T)\}, \tag{1.2}$$

where $\|z\|_2 := \left(|z_1|^2 + \cdots + |z_m|^2\right)^{\frac{1}{2}}$ is the Euclidean norm of $z = (z_1, \cdots, z_m)$. It is worth noting that $\sigma_l(T) \subseteq \sigma(T)$ and $r_l(T) \leq r(T)$.

Recall that a commuting m-tuple $T = (T_1, \cdots, T_m)$ on a Hilbert space $\mathcal{H}$ is said to be *joint subnormal* if there exist a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ and a commuting m-tuple $N = (N_1, \cdots, N_m)$ of normal operators N_i in $\mathcal{B}(\mathcal{K})$ such that

$$N_i h = T_i h \text{ for every } h \in \mathcal{H} \text{ and } 1 \leq i \leq m.$$

An m-tuple $S = (S_1, \cdots, S_m)$ of commuting operators S_i in $\mathcal{B}(\mathcal{H})$ is *joint hyponormal* if the $m \times m$ matrix $([T_j^*, T_i])_{1\leq i,j\leq m}$ is positive definite, where $[A, B]$ stands for the commutator $AB - BA$ of A and B. A joint subnormal tuple is always joint hyponormal [3].

Given a commuting m-tuple $T = (T_1, \cdots, T_m)$ on $\mathcal{H}$, we set

$$Q_T(X) := \sum_{i=1}^{m} T_i^* X T_i \ (X \in B(\mathcal{H})). \tag{1.3}$$