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Existence of weak solutions to steady compressible magnetohydrodynamic equations

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ABSTRACT

The three-dimensional equations of steady compressible magnetohydrodynamic non-isentropic flows are considered. A boundary value problem for the velocity and the temperature is studied in a bounded domain with arbitrarily large data. The existence of weak solutions to these equations is established under the assumption that the pressure $P(\rho, \theta) = c_1\rho^\gamma + c_2\rho\theta$ for $\gamma > \frac{7}{3}$.

1. Introduction and main result

Magnetohydrodynamics (MHD) [5] concerns the motion of conducting fluids in an electromagnetic field. The dynamic motion of the fluid and the magnetic field interact strongly on each other. It has a very broad range of applications. It is of importance in connection with many engineering problems, such as sustained plasma confinement for controlled thermonuclear fusion, liquid-mental cooling of nuclear reactors, and electromagnetic casting of metals. It also finds applications in geophysics and astronomy, where one prominent example is the so-called dynamo problem, that is, the question of the origin of the Earth's magnetic field in its liquid metal core.

Due to their practical relevance, MHD problems have long been the subject of intense research. The system of non-isentropic steady compressible magnetohydrodynamic equations in a three dimensional bounded domain Ω can be read as follows:

$$\mathrm{div}(\rho u) = 0, \tag{1.1}$$

$$\mathrm{div}(\rho u \otimes u) - \mathrm{div}\mathbb{S}(u) + \nabla P = \mu_0 \mathrm{rot} H \times H + \rho f, \tag{1.2}$$

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$$\operatorname{div}(Eu) = \rho f \cdot u - \operatorname{div}(Pu) + \operatorname{div}(\mathbb{S}u) + \operatorname{div}(\kappa(\theta)\nabla\theta) + \operatorname{div}[\mu_0(u \times H) \times H - \frac{1}{\sigma}\operatorname{rot}H \times H], \tag{1.3}$$

$$\frac{1}{\sigma}\Delta H = -\mu_0 \operatorname{rot}(u \times H), \quad \operatorname{div}H = 0, \tag{1.4}$$

where $\rho, u, P = c_1\rho^\gamma + c_2\rho\theta$, $c_1, c_2 > 0$ and H are density, velocity, pressure and magnetic field, respectively. $f \in C(\Omega)$ is a given vector field, and $\mathbb{S}(u) = \mu(\nabla u + \nabla u^\intercal) + \lambda \operatorname{div} u I$ is a viscous stress tensor. $\kappa(\theta) = c_3(1+\theta^m)$ with c_3, $m > 0$, the total energy $E = E(\rho, u, \theta) = \frac{1}{2}\rho|u|^2 + \rho e(\rho, \theta)$, and the internal energy e depends on the temperature and the density. For simplicity, we assume $e(\rho, \theta) = c_v\theta + \frac{1}{\gamma-1}\rho^{\gamma-1}$ with $c_v > 0$.

System (1.1)–(1.4) is equipped with the boundary conditions:

$$u \cdot n = 0, \quad \beta u \cdot \tau_k + (1-\beta)(\mathbb{S}n) \cdot \tau_k = 0, \tag{1.5}$$

$$(1+\theta^m)\frac{\partial\theta}{\partial n} + L(\theta)(\theta - \theta_0) = 0, \tag{1.6}$$

$$H \cdot n = 0, \quad \operatorname{rot}H \times n = 0, \tag{1.7}$$

where τ_k is the tangent vector to $\partial\Omega$, $\beta \in [0, 1]$. We will treat separately three cases: $\beta = 1$ which corresponds to the homogeneous Dirichlet condition $u = 0$ at $\partial\Omega$, $\beta = 0$ which corresponds to the total slip and requires additional assumptions on the geometry of Ω, and $\beta \in (0, 1)$. For the Navier boundary conditions (the last two cases), we denote $\alpha = \frac{\beta}{\beta-1} \geq 0$. Concerning the temperature, we assume $\theta_0(x) \geq c_0$ is a positive smooth function at the boundary and $L(\theta) = c_4(1 + \theta^l)$ with $l > 0$ and $c_4 > 0$. To avoid artificial technical computations, we restrict ourselves to the case $m = l + 1$.

When there is no electromagnetic field, the above governing system reduces to the compressible Navier–Stokes equations. The steady compressible Navier–Stokes equations for barotropic gases with arbitrarily large data were for the first time considered in the book [11], where the existence of weak solutions was established for $\gamma > 1$ $(N = 2)$ and $\gamma \geq \frac{5}{3}$ $(N = 3)$. The result was improved up to $\gamma > \frac{3}{2}$ in [9], however only for a potential force with a non-potential nonvolume force. Improvements, allowing γ slightly less than $\frac{5}{3}$ even for the nonpotential volume force, can be found in [1]. Further progress, allowing $\gamma \geq \frac{4}{3}$ $(N = 3)$ and $\gamma = 1$ $(N = 2)$ for the Dirichlet boundary conditions, can be found in [2,3]. Similar results, also for other boundary conditions, can be found in [4]. When $H \equiv 0$, problem (1.1)–(1.7) was considered in [7,8] where the existence of a weak solution was shown provided $\gamma > \frac{7}{3}$. To simplify we put $c_1 = c_2 = c_3 = c_4 = c_v = 1$.

Our main results are as follows:

Theorem 1.1. *Let $\beta \in [0, 1)$, $\Omega \in C^2$ be a bounded domain in R^3 which is not axially symmetric if $\beta = 0$. Let $m = l+1 > \frac{3\gamma-1}{3\gamma-7}$, $\gamma \in (\frac{7}{3}, 3]$, $f \in L^\infty(\Omega)$ and $M > 0$. Then there exists a weak solution to system (1.1)–(1.7) such that*

$$\rho \in L^{3\gamma-1}(\Omega), \quad \int\limits_\Omega \rho dx = M, \quad u \in H^1(\Omega), \quad H \in H^1(\Omega), \quad \theta \in H^1(\Omega) \cap L^{3m}(\Omega). \tag{1.8}$$

Theorem 1.2. *Let $\beta = 1$, the domain $\Omega \in C^2$, and let $m = l + 1 > \frac{3\gamma-1}{3\gamma-7}, \gamma \in (\frac{7}{3}, 3]$. Let $f \in L^\infty(\Omega)$ and $M > 0$. Then there exists a weak solution to system (1.1)–(1.7) such that*

$$\rho \in L^{s(\gamma)}(\Omega), \quad \int\limits_\Omega \rho dx = M, \quad u \in H^1(\Omega), \quad H \in H^1(\Omega), \quad \theta \in W^{1,r}(\Omega) \cap L^{3m}(\Omega), \tag{1.9}$$

where $s(\gamma) = \min\{3(\gamma - 1), 2\gamma\}$ and $r = \min\{2, \frac{3m}{m+1}\}$.

The solutions appeared in Theorems 1.1 and 1.2 are meant in the following sense.