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Darboux transformation and exact multisolitons of $\mathbb{C}P^N$ nonlinear sigma model

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ABSTRACT

Exact multisolitons of $\mathbb{C}P^N$ sigma model are obtained by using Darboux transformation in terms of quasideterminant Darboux matrix. The multisoliton solutions are shown to be expressed in terms of quasideterminants of Gelfand and Retakh (1991) [10]. The method of quasideterminant Darboux matrix has been shown to be related with the dressing method that has already been used to compute iteratively exact multisolitons of the $\mathbb{C}P^N$ sigma model by means of nonlinear superposition principle of Bäcklund transformations. Explicit expressions of one, two and three soliton solutions of $\mathbb{C}P^1$ model have been computed by using properties of quasideterminants.

1. Introduction

Two-dimensional $\mathbb{C}P^N$ sigma model is an important example of integrable field theories. It belongs to a general class of two-dimensional integrable field theories known as symmetric space nonlinear sigma models. It is a nonlinear sigma model with target space $\mathbb{C}P^N = SU(N+1)/S[U(N) \times U(1)]$. The nonlinear sigma models share many features of four dimensional nonabelian gauge theories such as conformal invariance, confinement, asymptotic freedom, existence of topological charge taking integer values, etc. The classical and quantum integrability of nonlinear sigma model has been investigated in many articles (see e.g. [4,14,1–3,5,7,6,15,29,20,22,21,31,16]). More recently, the $\mathbb{C}P^N$ sigma models have also been studied in the context of integrability in string theory on AdS spaces and in the construction of giant magnon solutions by using the well-known soliton solution generating techniques of integrable systems (see e.g. [23,32,24,25,12] and references therein).

As a classical integrable field theory, the $\mathbb{C}P^N$ model possesses Lax pair and exhibits existence of an infinite number of local as well as nonlocal conservation laws. The solution generating method of Bäcklund

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transformation of the $\mathbb{C}P^N$ model has been investigated in [29,20,22,21,31]. In the previous studies the solutions of the $\mathbb{C}P^N$ model were generated by iteratively using Bäcklund transformations. The Riccati equations and the corresponding Bäcklund transformations were used to develop a nonlinear superposition principle resulting in dressing method of generating multisoliton solutions of the model. The purpose of this paper is to study the multisoliton solutions of the $\mathbb{C}P^N$ model by using the Darboux transformation where it is expressed in terms of quasideterminant Darboux matrix and the solutions are expressed in terms of quasideterminants. The method of quasideterminant Darboux matrix has been shown to be related to the dressing method which has already been employed to generate multi-soliton solutions of the $\mathbb{C}P^N$ sigma model. We also obtain explicit expressions of one, two and three soliton solutions of $\mathbb{C}P^1$ model by using the properties of quasideterminants.

The $\mathbb{C}P^N$ sigma model involves maps from $\mathbb{R}^{1,1}$ to $\mathbb{C}P^N$ i.e. $z : \mathbb{R}^{1,1} \longrightarrow \mathbb{C}P^N$,

$$x^{\pm} = \frac{1}{2}(t \pm x) \longrightarrow z = (z^1, \cdots, z^{N+1}) \in \mathbb{C}^{N+1},$$

where the homogeneous coordinates $z = (z^1, \cdots, z^{N+1})$ obey $z \mapsto z' = \lambda z$ for $\lambda \neq 0$. The $\mathbb{C}P^N$ sigma model is a locally $U(1)$ invariant field theory defined by the Lagrangian density[1]

$$\mathcal{L} = \frac{1}{4}(D_+ z)^{\dagger}.D_- z, \tag{1.1}$$

where $\dagger$ denotes the hermitian conjugation and the projective invariance implies

$$|z|^2 = (z^{\dagger}.z) = \sum_{i=1}^{N+1} {z^i}^{\dagger} z^i = 1,$$

and $D_{\pm}$ are the $U(1)$ covariant derivatives which act on $z : \mathbb{R}^{1,1} \longrightarrow \mathbb{C}P^N$ as

$$D_{\pm} z = \partial_{\pm} z - z A_{\pm},$$

where $A_{\pm} = \left(z^{\dagger}.\partial_{\pm} z\right)$ are the components of the $U(1)$ gauge field associated with the $U(1)$ gauge symmetry. The Euler–Lagrange equations of $\mathbb{C}P^N$ model resulting from the Lagrangian are

$$D_+ D_- z + \left((D_+ z)^{\dagger}.(D_- z)\right) z = 0. \tag{1.2}$$

For $N = 1$, we have the simplest sigma model, the $\mathbb{C}P^1$ model which is equivalent to the $O(3)$ sigma model. The equivalence is established by taking $\phi^i = z^{\dagger}\sigma_i z$, where σ_i are Pauli spin matrices. In this case

$$\begin{aligned} \mathcal{L} = \frac{1}{4}(D_+ z)^{\dagger}.D_- z &\to \frac{1}{4}\partial_+ \phi_i \partial_- \phi_i, \\ \text{and} \qquad |z|^2 = 1 &\to \phi_i \phi_i = 1. \end{aligned} \tag{1.3}$$

For our purpose we work with a $U(N)$ valued field $g = (z, Y)$ where $z \in \mathbb{C}^{N+1}$ and Y is a complex $N \times N+1$ matrix such that g is a moving orthogonal frame of $\mathbb{C}^{N+1}$. Now the Euler–Lagrange equations are expressed in terms of the $U(1) \times U(N)$ gauge fields

$$\widetilde{A}_{\pm} = \begin{pmatrix} z^{\dagger}\partial_{\pm} z & 0 \\ 0 & Y^{\dagger}\partial_{\pm} Y \end{pmatrix}, \tag{1.4}$$

[1] The spacetime conventions are such that light-cone coordinates $x^{\pm}$ are related to the orthonormal coordinates by $x^{\pm} = \frac{1}{2}(t \pm x)$ with derivatives $\partial_{\pm} = \frac{1}{2}(\partial_t \pm \partial_x)$.