J. Math. Anal. Appl. 450 (2017) 81–111

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

The absolute continuity of convolutions of orbital measures in symmetric spaces ☆

Sanjiv Kumar Gupta [a], Kathryn E. Hare [b,*]

[a] Dept. of Mathematics and Statistics, Sultan Qaboos University, P.O. Box 36, Al Khodh 123, Oman
[b] Dept. of Pure Mathematics, University of Waterloo, Waterloo, ON, N2L 3G1, Canada

ARTICLE INFO

Article history:
Received 8 June 2016
Available online 16 January 2017
Submitted by M. Peloso

Keywords:
Orbital measure
Symmetric space
Double coset
Absolute continuity

ABSTRACT

We characterize the absolute continuity of convolution products of orbital measures on the classical, irreducible Riemannian symmetric spaces G/K of Cartan type III, where G is a non-compact, connected Lie group and K is a compact, connected subgroup. By the orbital measures, we mean the uniform measures supported on the double cosets, KzK, in G. The characterization can be expressed in terms of dimensions of eigenspaces or combinatorial properties of the annihilating roots of the elements z. A consequence of our work is to show that the convolution product of any rank G/K, continuous, K-bi-invariant measures is absolutely continuous in any of these symmetric spaces, other than those whose restricted root system is type A_n or D_3, when rank G/K +1 is needed.

1. Introduction

In this paper, we study the smoothness properties of K-bi-invariant measures on the irreducible Riemannian symmetric spaces G/K, where G is a connected Lie group and K is a compact, connected subgroup fixed by a Cartan involution of G. Inspired by earlier work of Dunkl [6] on zonal measures on spheres, Ragozin in [29] and [30] showed that the convolution of any $\dim G/K$, continuous, K-bi-invariant measures on G was absolutely continuous with respect to the Haar measure on G. This was improved by Graczyk and Sawyer, who in [10] showed that if G was non-compact and $n = \text{rank } G/K$, then any $n+1$ convolutions of such measures is absolutely continuous and that this is sharp for the symmetric spaces whose restricted root system was type A_n. They conjectured that $n+1$ was always sharp. One consequence of our work is to show this conjecture is false. In fact, for all the classical, non-compact symmetric spaces of rank n, other than those whose restricted root system is type A_n, the convolution product of any n K-bi-invariant, continuous measures is absolutely continuous and this is sharp.

☆ This research is supported in part by NSERC #44597. The first author thanks the University of Waterloo for their hospitality when some of this research was done.

* Corresponding author.
E-mail addresses: gupta@squ.edu.om (S.K. Gupta), kehare@uwaterloo.ca (K.E. Hare).

http://dx.doi.org/10.1016/j.jmaa.2017.01.027

We obtain this result by studying a particular class of examples of K-bi-invariant, continuous measures, the so-called orbital measures

$$\nu_z = m_K * \delta_z * m_K,$$

where m_K is the Haar measure on K. These are the uniform measures supported on the double cosets KzK in G. They are purely singular, probability measures. The main accomplishment of this paper is to characterize the L-tuples $(z_1, ..., z_L)$ such that $\nu_{z_1} * \cdots * \nu_{z_L}$ is absolutely continuous for the classical symmetric spaces of Cartan type III.

The convolution product $\nu_{z_1} * \cdots * \nu_{z_L}$ is supported on the product of double cosets $Kz_1Kz_2 \cdots Kz_LK$ and hence if the convolution product is absolutely continuous then the product of double cosets has positive Haar measure in G. In fact, if the convolution product is absolutely continuous, the product of double cosets has non-empty interior and the converse is true, as well, thus we also characterize which products of double cosets have non-empty interior.

Given any $z_j \in G$ there is some $Z_j \in \mathfrak{g}$, the Lie algebra of G, such that $z_j = \exp Z_j$. Our characterization is in terms of combinatorial properties of the set of annihilating roots of the elements Z_j. It also can be expressed in terms of the dimensions of the largest eigenspaces when we view the Z_j as matrices in the classical Lie algebras.

In a series of papers, (see [11–13] and the references cited therein), Graczyk and Sawyer found a characterization for the absolute continuity of $\nu_x * \nu_y$ for certain of the type III (mainly) classical symmetric spaces. A survey of their work can be found in [14]. Our approach was inspired by their work, but is more abstract and relies heavily upon combinatorial properties of the root systems and root spaces of Lie algebras.

The Cartan involution also gives rise to a decomposition of the Lie algebra as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Take a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ and put $A = \exp \mathfrak{a}$. Then $G = KAK$, thus we can always assume $z \in A$ when studying orbital measures. Closely related to the K-bi-invariant orbital measures supported on double cosets in G are the K-invariant, uniform measures, μ_Z, supported on the orbits under the $Ad(K)$ action of elements $Z \in \mathfrak{a}$. It is known ([1]) that $\mu_{Z_1} * \cdots * \mu_{Z_L}$ is absolutely continuous with respect to Lebesgue measure on $\mathfrak{p}$ if and only if $\nu_{z_1} * \cdots * \nu_{z_L}$ is absolutely continuous on G when $z_j = \exp Z_j$, and this is also equivalent to the sum of the $Ad(K)$ orbits generated by the Z_j having non-empty interior.

The absolute continuity of convolution products of orbital measures is connected with questions about spherical functions $\phi_\lambda(\exp X)$ where λ is a complex-valued linear form on $\mathfrak{a}$ and $X \in \mathfrak{a}$. This is the spherical Fourier transform of the orbital measure $\nu_{\exp X}$. The product formula states that

$$\phi_\lambda(e^{Z_1}) \cdots \phi_\lambda(e^{Z_L}) = \int_G \phi_\lambda d\nu_{z_1} * \cdots * \nu_{z_L},$$

hence the absolute continuity of convolutions of orbital measures gives a formula for a product of spherical functions.

An example of a compact, symmetric space is $(G \times G)/\Delta(G)$ where G is a compact, connected, simple Lie group and $\Delta(G) = \{(g, g) : g \in G\} \simeq G$. These are the symmetric spaces of Cartan type II. In this case, the orbital measure ν_z for $z = (g, g)$, supported on the double coset $\Delta(G)z\Delta(G)$, can be identified with the uniform measure supported on the conjugacy class in G containing the element g. The $\Delta(G)$-invariant measure, μ_Z for $Z \in \mathfrak{g}$, can be identified with the measure on the compact Lie algebra $\mathfrak{g}$ that is uniformly distributed on the adjoint orbit in $\mathfrak{g}$ containing Z. These symmetric spaces are dual to the non-compact, symmetric spaces of Cartan type IV (see [1,22]) and it can be seen from [1] that the absolute continuity problem for Cartan type IV symmetric spaces can be deduced from the analogous problem for the corresponding G-invariant orbital measures μ_Z for $Z \in \mathfrak{g}$, from the (dual) Cartan type II spaces.

In a series of papers, the authors (with various coauthors) studied the absolute continuity problem for orbital measures in the compact setting. The sharp exponent $n = n(Z)$ (or $n(z)$) with the property that