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Initial-boundary value problems for the equations of the global atmospheric electric circuit

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ABSTRACT

Time-dependent and steady-state problems for the equations of the global atmospheric electric circuit are studied in the quasi-stationary approximation. Maxwell's equations in an anisotropic medium are written in terms of the electric potential; several types of boundary conditions (motivated by applications) are considered, including the analogues of the classical Dirichlet, Neumann and mixed conditions, as well as a non-standard condition relating the potential and normal current density at symmetric points on one of the boundary components. In each case the variational formulation of the problem is derived and is shown to be well-posed. Also, certain stabilisation theorems are proved for the solution of time-dependent problem in case the main parameters do not depend on time.

1. Introduction

The concept of the global electric circuit (GEC) is fundamentally important for the theoretical understanding of atmospheric electricity (see, e.g., [25,31,32]). The term 'global electric circuit' refers to the electric current distribution in the Earth's atmosphere; this distribution includes, for instance, lightning currents, precipitation currents and corona discharge currents, but its most important constituent is the so-called quasi-stationary current, which flows continuously and, according to the hypothesis originally proposed by Wilson [33,34], is maintained by permanent charge separation in thunderstorms and other electrified clouds. Roughly speaking, this current flows upwards in thunderstorm regions and flows downwards in fair-weather regions, the circuit being completed by the highly conductive Earth's surface and lower ionosphere.

The GEC modelling has been given much attention over the past few decades [4,13–15,19]. The majority of the existing GEC models are aimed at finding the distributions of the quasi-stationary current density

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and the electric field in the atmosphere, given the GEC generators, which correspond to electrified clouds. Usually these generators are regarded as a certain distribution of the so-called source current density, which enters into Ohm's law along with the conduction current density.

In order to study the GEC from a mathematical perspective and to further develop the existing GEC models, one must deal with Maxwell's equations. The mathematical foundations of the electromagnetic theory have been established in the last few decades; the well-posedness of various steady-state and time-dependent problems for Maxwell's equations has been studied in detail, allowing for inhomogeneous media, inhomogeneous boundary conditions and domains with complicated topology (e.g., [1,2,5,7,11,24,26]). However, particular applications may lead to new problems, which, owing to certain peculiarities, cannot be reduced to the problems described in the literature and which have yet to be considered in detail.

In this article we consider the time evolution of the GEC, described by Maxwell's equations within the quasi-stationary approximation; in this approximation the electric field may be described in terms of a scalar potential. The most important aspects of this problem include a broad variety of possible boundary conditions (corresponding to different physical problems), the non-trivial topology of the Earth's atmosphere and the fact that the main equation for the electric potential is not resolved with respect to the time derivative. The variational problems described in this article provide the basis for novel numerical models of the GEC and GEC-related physical problems.

2. Motivation

We shall start with a preliminary discussion of the equations describing the atmospheric GEC. In this discussion we do not concern ourselves with rigorous formulations, as it is intended only to explain our motivation and the origin of the problem. Therefore we assume, for the moment, that all the functions, domains and boundary surfaces that appear in our reasoning are sufficiently smooth.

We suppose that the atmosphere occupies the region Ω, whose boundary consists of the Earth's surface Γ_1 and a surface Γ_2 representing the lower limit of the ionosphere, the latter encompassing the former (see Fig. 1a). We also suppose that both the dielectric permittivity and the magnetic permeability of the atmosphere are equal to 1, in which case non-stationary Maxwell's equations read as follows[1]:

$$\operatorname{curl}\mathbf{H} = \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} + \frac{4\pi}{c}\mathbf{j}, \tag{2.1}$$

$$\operatorname{curl}\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t}, \tag{2.2}$$

$$\operatorname{div}\mathbf{H} = 0, \tag{2.3}$$

$$\operatorname{div}\mathbf{E} = 4\pi\rho, \tag{2.4}$$

where $\mathbf{E}(t,\mathbf{x})$ is the electric field, $\mathbf{H}(t,\mathbf{x})$ is the magnetic field, $\mathbf{j}(t,\mathbf{x})$ is the current density, $\rho(t,\mathbf{x})$ is the charge density, $\mathbf{x}$ denotes the spatial coordinates, t stands for time and c stands for the speed of light. These equations must be supplemented with Ohm's law

$$\mathbf{j} = \sigma\mathbf{E} + \mathbf{j}^{\mathrm{s}}, \tag{2.5}$$

where $\sigma(t,\mathbf{x})$ is the conductivity and $\mathbf{j}^{\mathrm{s}}(t,\mathbf{x})$ represents the source current density, as well as with suitable initial and boundary conditions. We regard thunderstorms as distributed current sources and suppose that the source current density is non-zero only within thunderclouds and other electrified clouds. In other words, at every moment in time the positions of thunderstorms correspond to the spatial distribution of $\mathbf{j}^{\mathrm{s}}$ (see Fig. 1a).

[1] Hereafter we use the Gaussian unit system.