J. Math. Anal. Appl. 450 (2017) 197–228

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Distance to the line in the Heston model

Archil Gulisashvili

Department of Mathematics, Ohio University, Athens, OH 45701, USA

A R T I C L E I N F O

Article history:
Received 13 November 2014
Available online 16 January 2017
Submitted by P. Yao

This article is dedicated to the memory of Peter Laurence

Keywords:
Heston manifold
Distance to the line
Level sets
Minimization problems
Implied volatility

A B S T R A C T

The main object of study in the paper is the distance from a point to a line in the Riemannian manifold associated with the Heston model. We reduce the problem of computing such a distance to certain minimization problems for functions of one variable over finite intervals. One of the main ideas in this paper is to use a new system of coordinates in the Heston manifold and the level sets associated with this system. In the case of a vertical line, the formulas for the distance to the line are rather simple. For slanted lines, the formulas are more complicated, and a more subtle analysis of the level sets intersecting the given line is needed. We also find simple formulas for the Heston distance from a point to a level set. As a natural application, we use the formulas obtained in the present paper in the study of the small maturity limit of the implied volatility in the Heston model.

1. Introduction

In this paper, we study a special Riemannian manifold. We call it the Heston manifold because it is intimately related to the Heston model of financial mathematics.

The Heston model is one of the classical stock price models with stochastic volatility. The stock price process S and the variance process V in the Heston model satisfy the following system of stochastic differential equations:

$$\begin{cases} dS_t = rS_t dt + \sqrt{V_t} S_t dW_t \\ dV_t = (a - bV_t)dt + c\sqrt{V_t} dZ_t, \end{cases} \tag{1}$$

where $a \geq 0$, $b \geq 0$, $c > 0$, and $r \geq 0$ is the interest rate. In (1), W and Z are correlated standard Brownian motions such that $d\langle W, Z\rangle_t = \rho dt$ with $\rho \in (-1, 1)$. The initial condition for the process S is denoted by s_0. The Heston model was introduced in [15]. We refer the interested reader to [10,11,14,19] for more information on the Heston model.

E-mail address: gulisash@ohio.edu.

http://dx.doi.org/10.1016/j.jmaa.2017.01.014
0022-247X/

For the sake of simplicity, we assume throughout the paper that the interest rate r is equal to zero. Then, the log-price process X and the variance process V in the Heston model satisfy the following system of stochastic differential equations:

$$\begin{cases} dX_t = -\frac{1}{2}V_t dt + \sqrt{V_t}dW_t \\ dV_t = (a - bV_t)dt + c\sqrt{V_t}dZ_t. \end{cases}$$

The state space for the process (X, V) is the closed half-plane $\mathcal{H} = \left\{(x, v) \in \mathbb{R}^2 : v \geq 0\right\}$. The initial condition for the two-dimensional process (X, V) will be denoted by (x_0, v_0).

The Riemannian metric form associated with the Heston model, for which $c = 1$ and $\rho = 0$, is defined on the interior $\mathcal{H}^\circ$ of the closed half-plane $\mathcal{H}$ as follows: $ds^2 = v^{-1}\left(dx^2 + dv^2\right)$. This form generates the Riemannian distance d_H on $\mathcal{H}$. We call the open half-plane $\mathcal{H}^\circ$, equipped with the metric form defined above, the Heston Riemannian manifold (see [13] for more details). The line $\{(x, v) : v = 0\}$ is the boundary of the Heston manifold, and the manifold is incomplete.

Remark 1. Riemannian metrics similar to the Heston metric also appear in other fields of mathematics. For example, P. Daskalopoulos and R. Hamilton used the Riemannian metric in the right half-plane, defined by $ds^2 = (2x)^{-1}\left(dx^2 + dy^2\right)$, to study the regularity of the interface of the evolution p-Laplacian equation (see [4]) and the porous medium equation (see [3]). Daskalopoulos and Hamilton call this metric the cycloidal metric, since all the geodesics of this metric can be obtained from the standard cycloid curve by translation and dilation, or are horizontal lines (see Proposition I.2.1 in [3]).

Methods of mathematical analysis and differential geometry found numerous applications in quantitative finance. A good source of information about such applications is the book [14]. In [13], the author of the present paper and P. Laurence found the following formula for the Heston Riemannian distance between any two points $(x_0, v_0) \in \mathcal{H}$ and $(x_1, v_1) \in \mathcal{H}$ in the case, where at least one of the points is not on the boundary:

$$d_H\left((x_0, v_0), (x_1, v_1)\right) = \frac{\delta}{\sin\left(\frac{\delta}{2}\right)}\sqrt{v_1 + v_0 - 2\sqrt{v_1 v_0}\cos\left(\frac{\delta}{2}\right)}, \tag{2}$$

where $\delta = \delta((x_0, v_0), (x_1, v_1))$ is the unique solution to the equation

$$\frac{(v_1 + v_0)\left(\delta - \sin(\delta)\right) - 2\sqrt{v_1 v_0}\left(\delta\cos\left(\frac{\delta}{2}\right) - 2\sin\left(\frac{\delta}{2}\right)\right)}{2\sin^2\left(\frac{\delta}{2}\right)} = x_1 - x_0, \tag{3}$$

satisfying the condition $-2\pi < \delta < 2\pi$.

One of the main objectives in the present paper is to study the distance from a point to a line in the Heston manifold. Fix real numbers γ and β, and denote by $L_{\beta,\gamma}$ the line in the upper half-plane $\mathcal{H}$ given by $\{(x, v) \in \mathcal{H} : x = \beta + \gamma v,\ v \geq 0\}$. The symbol $\widehat{D}_{\beta,\gamma}$ will stand for the distance from the model point $(0, 1) \in \mathcal{H}$ to the line $L_{\beta,\gamma}$ in the uncorrelated Heston model ($\rho = 0$) with $c = 1$. More precisely,

$$\widehat{D}_{\beta,\gamma} = \inf_{v \geq 0}\left\{d_H((0, 1), (\beta + \gamma v, v))\right\}. \tag{4}$$

It is worth mentioning here that our choice of the point $(0, 1)$ in the Heston manifold and the parameters $\rho = 0$ and $c = 1$ does not restrict the generality in the distance to the line problem (see Remark 2 in Section 2).