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Approximation of functions by a new family of generalized Bernstein operators

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ABSTRACT

The main object of this paper is to construct a new generalization of the Bernstein operator, depending on a non-negative real parameter. We investigate some elementary properties of this operator, such as end point interpolation, linearity and positivity, etc. By using these generating operators, we provide another proof of the Weierstrass Approximation Theorem. We give the detailed proofs to the rate of convergence and Voronovskaja type asymptotic estimate formula for the operators. Moreover, an upper bound for the error is obtained in terms of the usual modulus of continuity. Shape preserving properties of the generalized Bernstein operators are also studied. It is proved that monotonic or convex functions produce monotonic or convex generalized Bernstein polynomials.

1. Introduction

The approximation of real-valued continuous functions by simpler class of functions, say, the class of algebraic polynomials, has attracted the attention of thousands researchers during the last two centuries. A fundamental result in development of functions approximation theory is known as Weierstrass Approximation Theorem, which states that for any continuous function $f(x)$ defined on $[a,b]$ and any $\varepsilon > 0$, there corresponds an algebraic polynomial $P(x)$ with real coefficients, such that $|f(x) - P(x)| < \varepsilon$ for all $x \in [a,b]$. In other words, polynomials can uniformly approximate any function that is merely continuous over a closed interval.

There are several proofs of this fundamental theorem, beginning with that given by K. Weierstrass [23] in 1885. But the first proof of this approximation theorem was long and complicated, which provoked many famous mathematicians to find simpler and more instructive proofs. One of the simplest and most elegant ways to prove this theorem was given by S.N. Bernstein [3] in 1912. On this occasion the following very

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well-known Bernstein operators were constructed

$$B_n(f;x) = \sum_{i=0}^{n} f\left(\frac{i}{n}\right) p_{n,i}(x),$$

for any continuous function $f(x)$ defined on $C[0,1]$, where

$$p_{n,i}(x) = \binom{n}{i} x^i (1-x)^{n-i}$$

is the classical Bernstein basis function. The Bernstein operators provide a simple and constructive proof to the Weierstrass Approximation Theorem for the case $C[0,1]$. With the help of the function

$$[0,1] \to [a,b], \quad t = a(1-x) + bx,$$

the result extends to $C[a,b]$. These operators are the most studied linear positive operators and were generalized and modified in a great number of variants. The advantages of the Bernstein operators consist in their simplicity, and on their superior properties of approximation.

Yet there is a more general approach to the theorem of Weierstrass involving a sequence of positive linear operators. The importance of the Bernstein polynomials leads to the discovery of their numerous generalizations, one of which is the q-Bernstein polynomials. Lupaş [12] introduced in 1987 a q-type of the Bernstein operators and a decade later another generalization of these operators based on q-integers was introduced by Philips [19]. In recent years, many researchers [10,1,15,8,7] focused their attention on the study of generalized version in q-calculus of well-known linear and positive operators. To approximate Lebesgue integrable functions on the interval $[0,1]$, Durrmeyer [6] introduced the integral modification of the well-known Bernstein polynomials. The q-generalization of these operators was introduced by Gupta and Wang [9]. The study of these famous operators is extended and revisited year after year in many directions. In 2013, Mishra and Patel [17] introduced a kind of Stancu generalization of q-Durrmeyer operators, and one year later Vishnu [18] studied the convergence behavior of the q-analog of the Stancu generalization of Durrmeyer operators. Mishra studied hypergeometric representation, simultaneous approximation and inverse result for Baskakov–Durrmeyer–Stancu type operators [14,16,13]. Cárdenas-Morales [5] presented the new sequence of linear Bernstein-type operators and Başcanbaz-Tunca [2] discussed bivariate extension of this Bernstein-type operators. Szabados [21] considered a special case of the modification of Lagrange interpolation due to Bernstein. Compared to Lagrange interpolation, these operators interpolate at less points, but they converge for all continuous functions in case of the Chebyshev nodes. There are many other generalizations of the Bernstein operators which have been investigated by many researchers.

The present work is organized as follows. In the first section, we give definition of a new family of the generalized Bernstein operators and their certain elementary properties, which play an important role in the theory of uniform approximation of functions. In the second section, the main purpose is to study some important results concerning uniform convergence and estimates of the new linear positive operators, which are direct applications of the properties and formulas recalled in the first section. Finally, we study the shape preserving properties of the new operator: It is shown that if $f(x)$ is increasing (or decreasing) on $[0,1]$, then the new operator is also increasing (or decreasing) on $[0,1]$; and if $f(x)$ is convex (or concave), then so is the operator.

2. The new generalized Bernstein operators

Bernstein operators constitute a powerful tool allowing one to replace many inconvenient calculations performed for continuous functions by more friendly calculations on approximating polynomials. Since the