J. Math. Anal. Appl. ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Existence and uniqueness of traveling waves for a reaction–diffusion model with general response functions

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A R T I C L E I N F O

Article history:
Received 6 July 2016
Available online xxxx
Submitted by G. Chen

Keywords:
Traveling wave
Reaction–diffusion equation
Schauder fixed point theorem
Minimum wave speed

A B S T R A C T

The traveling wave solutions of a reaction–diffusion model with general functional responses are considered in this paper. A necessary and sufficient condition is established for the existence of traveling wave solutions. Moreover, the minimum speed, asymptotic behavior and uniqueness of traveling wave solutions are also obtained in relation to the interaction parameters.

1. Introduction

In this paper, we study the following prey–predator reaction–diffusion system

$$\begin{aligned} \frac{\partial S}{\partial t} &= d_0 \frac{\partial^2 S}{\partial x^2} - \nu \frac{\partial S}{\partial x} - \frac{1}{\gamma} f(S,u)u, \\ \frac{\partial u}{\partial t} &= d_1 \frac{\partial^2 u}{\partial x^2} - \nu \frac{\partial u}{\partial x} + f(S,u)u - ku, \end{aligned} \tag{1.1}$$

where $S(x,t)$ and $u(x,t)$ are the concentrations of nutrient and microbial population at position x and time t, respectively. The parameter $\nu \geq 0$ is the flow velocity; $k > 0$ is the mortality rate; $\gamma > 0$ is the yield rate; $d_i > 0 (i = 0, 1)$ are the diffusion rate. The nutrient uptake function

$$f(S,u) = \frac{S}{r + au + bS}, \tag{1.2}$$

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http://dx.doi.org/10.1016/j.jmaa.2017.01.017

where r, a, b are nonnegative constants with $r^2+a^2+b^2 \neq 0$. System (1.1) serves as a mathematical model to biological or chemical phenomena, such as bacterial growth on a limited substrate [12], an infection outbreak in a crowded population structure [3,4], a chemical reaction involving a limiting reactant [13,20], etc.

In this paper, we look for the traveling wave solutions of (1.1), which reflect an important phenomenon of wave propagation. A traveling wave solution of (1.1) is a special type of solutions with the form

$$S = S(z), \quad u = u(z), \quad z = x + ct,$$

where the constant c is the wave speed. It represents the transition process from the initial equilibrium state to another end equilibrium state. For each constant S^*, it is easy to see that (1.1) has an equilibrium $(S^*, 0)$. From the biological and mathematical points of view, we're only interested in the existence of a traveling wave solution which connects two different equilibria $(S^0, 0)$ and $(S_0, 0)$. For instance, in the infinitely long flow-reactor, S^0 represents the input concentration of nutrient at the left end of the reactor $(x = -\infty)$ and S_0 represents the concentration of washed-out nutrient at the right end $(x = \infty)$, where S_0 is an unknown constant. Moreover, one might expect $S_0 < S^0$ because the nutrient should be sufficient for growth upstream of the pulse and be depleted below the level at which bacteria can grow downstream of the pulse. Hence, it is reasonable to assume $S(z)$ and $u(z)$ satisfying the boundary conditions

$$S(-\infty) = S^0 > S(\infty) = S_0, \quad u(\pm\infty) = 0. \tag{1.3}$$

If predators compete directly for the available prey, the nutrient uptake function $f(S, u)$ is independent of u. In this case, the traveling wave solutions for the system (1.1) have been widely studied. By the application of singular perturbation method and center manifold theorem, the existence of traveling wave solutions of (1.1) and (1.3) with small d_0 is established in [12]. In [5], Huang investigated the existence of traveling waves of (1.1) and (1.3) with an arbitrary diffusion coefficient by the shooting method. The results show that there exists a minimum wave speed c^* such that (1.1) and (1.3) admit a traveling wave if and only if $c \geq c^*$. Furthermore, the uniqueness of traveling wave solutions of (1.1) and (1.3) are shown in [6]. Wang and Wu studied the existence and non-existence of traveling wave solutions of a bio-reactor model with stage-structure in [15] by the fixed point theorem on cone. The more detailed works on the existence and uniqueness of traveling wave solutions refer to [3,4,8,13,14,20] and references therein.

However, the predators growth should depend not only on the prey density but also on the predator density. To describe this phenomenon precisely, ratio-dependent uptake function and Beddington–DeAngelis uptake function are proposed, which are included in the general response function (1.2). In [17], by Schauder fixed point theorem and Laplace transform, the authors investigated the existence and non-existence of traveling wave solutions of (1.1)–(1.3) with the ratio-dependent uptake function. The existence, uniqueness, multiplicity and stability of the coexistence solutions of (1.1) with the Beddington–DeAngelis uptake function are studied in [11,16] by the application of degree theory in cones, bifurcation theory and perturbation technique. However, there are few works on the traveling wave solutions of (1.1) with the Beddington–DeAngelis uptake function.

The focus of this paper is to establish the existence, uniqueness and the critical speed of propagation for traveling wave solutions of (1.1)–(1.3). The main results can be stated as follows.

Theorem 1.1. *For given S^0 with $f(S^0, 0) > k$ and $c^* = 2\sqrt{d_1[f(S^0, 0) - k]} - \nu$, we have*

(i) *if $c \geq c^*$, then there is a unique nonnegative constant S_0 satisfying $f(S_0, 0) \leq k$ such that (1.1) exists a positive traveling wave solution $(S(x+ct), u(x+ct))$ connecting the equilibria $(S^0, 0)$ and $(S_0, 0)$. Moreover,*

$$k \int_{-\infty}^{\infty} u(s)ds = \gamma(c + \nu)(S^0 - S_0);$$