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Partially normal composition operators relevant to weighted directed trees

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This paper is dedicated to the memory of Professor Takayuki Furuta

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ABSTRACT

We characterize properties including p-hyponormality and p-paranormality for composition operators arising from measurable transformations on weighted directed trees, in terms of a test at each node v involving the masses at nodes in a neighborhood of nodes near v. Also constructed are certain graphs $\mathcal{E}$ "universal" for p-hyponormality in that the neighborhood of any node in any graph yielding a p-hyponormal composition operator is a certain limit of neighborhoods in $\mathcal{E}$. These results are applied to some examples with particularly regular graph structures.

1. Introduction and preliminaries

Let $\mathcal{H}$ be a separable, infinite dimensional complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. The gap between normal and hyponormal operators has been considered by many operator theorists, and this study continues. One property used in this study is p-hyponormality; an operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *p-hyponormal* if $(T^*T)^p \geq (TT^*)^p$, $p \in (0, \infty)$. If $p = 1$, T is *hyponormal* and if $p = \frac{1}{2}$, T is *semi-hyponormal* [15]. As well, T is said to be *∞-hyponormal* if it is p-hyponormal for all $p > 0$ [10]. According to the Löwner–Heinz inequality [15,5], every q-hyponormal operator is p-hyponormal for $p \leq q$. And T is *p-paranormal* if $\| \, |T|^p \, U \, |T|^p \, x\| \geq \| \, |T|^p \, x\|^2$ for all unit vectors $x \in \mathcal{H}$. In particular, 1-paranormality is referred to as paranormality. Every q-paranormal operator is p-paranormal for $q \leq p$. It

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is obvious that every p-hyponormal operator is p-paranormal, $p \in (0, \infty)$. Hence any p-hyponormal operator is q-paranormal for all $q \in (0, \infty)$.

Previously Burnap–Jung–Lambert discussed composition operators C_T on L^2 via conditional expectations in [2] and [1], in which they proved that the classes of p-hyponormal operators are distinct for each positive real number p and the p-paranormal operators classes are as well. They used conditional expectations to detect p-hyponormality of C_T, which will be main tool of this note. Also, Jung–Lim–Park constructed examples induced by some block matrix operators in [8], in which they proved that the classes of those operators are distinct with respect to each positive real number p. In [4] a model of a block matrix operator induced by two sequences was introduced and p-hyponormality was characterized, and distinctions obtained, via conditional expectation. This paper is a continuation of the study [4] of p-hyponormalities of composition operators on ℓ^2.

Here is some notation and terminology related to conditional expectation. Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space and let $T : X \to X$ be a nonsingular measurable transformation, i.e., $T^{-1}\mathcal{F} \subset \mathcal{F}$ and $\mu \circ T^{-1} \ll \mu$. We assume that the Radon–Nikodym derivative $h = d\mu \circ T^{-1}/d\mu$ is in L^∞. The composition operator C_T acting on $L^2 := L^2(X, \mathcal{F}, \mu)$ is defined by $C_T f = f \circ T$. The condition $h \in L^\infty$ assures that C_T is bounded. We consider here only bounded composition operators on L^2. We denote the conditional expectation of f with respect to $T^{-1}\mathcal{F}$ by $Ef = E(f|T^{-1}\mathcal{F})$. We recall some known results from [9,2,6], which will be used frequently through this paper. The interested reader can find a more extensive list of properties for conditional expectations in [6] and [12]. Every $T^{-1}\mathcal{F}$ measurable function has the form $F \circ T$, where F is $\mathcal{F}$-measurable function. Note that $F \circ T = G \circ T$ if and only if $hF = hG$; in fact, $F \circ T \geq G \circ T$ if and only if $F\chi_S \geq G\chi_S$ where $S =$ support h and χ_S is the characteristic function of S [3]. It is known that $C_T^* f = h(Ef) \circ T^{-1}$ (the previous two properties show that this expression is well-defined) and $h \circ T > 0$ a.e. Also, it is well-known that $C_T^* C_T f = hf$ for $f \in L^2$ and $C_T C_T^* f = (h \circ T)Ef$ for $f \in L^2$. In particular, we will have need of the following special case: if $\mathcal{A}$ is the purely atomic σ-subalgebra of $\mathcal{F}$ generated by the partition of X into sets of positive measure $\{A_k\}_{k=0}^{\infty}$, then

$$E(f|\mathcal{A}) = \sum_{k=0}^{\infty} \frac{1}{\mu(A_k)} \left(\int\limits_{A_k} f(x) d\mu(x) \right) \chi_{A_k}. \tag{1.1}$$

The following known results will be crucial in this paper.

1° C_T is normal if and only if $T^{-1}\mathcal{F} = \mathcal{F}$ and $h = h \circ T$ [6].
2° C_T is quasinormal if and only if $h = h \circ T$ [14].
3° C_T is ∞-hyponormal if and only if $h \geq h \circ T$ [2].
4° C_T is p-hyponormal if and only if $h > 0$ and $E(1/h^p) \leq 1/(h^p \circ T)$, for $p \in (0, \infty)$ [2].
5° C_T is p-paranormal if and only if $E(h^p) \geq h^p \circ T$ [1].

The idea in [8] and [4] provides a good motivation to study composition operators on the usual Hardy space $\ell^2(V)$ defined by a node set in a weighted directed tree $\mathcal{G} = (V, E, \mu)$ (whose notation is in the next section). For a directed tree $\mathcal{G}$ with masses, a measurable transformation T can be defined on a node set V. A composition operator C_T can be defined by such a transformation T and can be analyzed using the results 1°–5°.

This paper consists of four sections. In Section 2, we introduce some fundamental definitions and properties from graph theory for our purposes. We define a weighted directed tree $\mathcal{G} = (V, E, \mu)$ which provides a measurable transformation T on $\mathcal{G}$, with associated composition operator C_T on $\ell^2(V)$. With those constructions we characterize normal, quasinormal, ∞-hyponormal, p-hyponormal, and p-paranormal composition operators C_T induced by such measurable transformations T on (V, μ). In Section 3, we consider all $C_T(\mathcal{G})$