J. Math. Anal. Appl. 450 (2017) 544–577

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Dynamical poroplasticity model – Existence theory for gradient type nonlinearities with Lipschitz perturbations

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ARTICLE INFO

Article history:
Received 1 July 2016
Available online 24 January 2017
Submitted by P. Yao

Keywords:
Biot model
Coercive approximation
Inelastic deformation
Gradient type operator
Poroplasticity

ABSTRACT

In this article we study existence theory to a non-coercive fully dynamic model of poroplasticity with the non-homogeneous boundary conditions where the constitutive function is a continuous element of class $\mathcal{LM}$ (it is a sum of a maximal monotone map G and a globally Lipschitz map l). Without any additional growth conditions we are able to prove the existence of a solution such that the inelastic constitutive equation is satisfied in the measure-valued sense. Moreover, if G is a gradient of a differentiable convex function, then there exists a solution such that the constitutive equation is satisfied almost everywhere.

1. Introduction

In this paper we study existence theory to a dynamic model for the poroplastic behaviour of soil. It has been introduced in 1993 by W. Ehlers in [16]. It can be used to describe porous media (brittle, granular) which are often saturated by some liquids or gases. Naturally, soil is a good example of such media.

The considered model consists of the dynamical Biot model of soil consolidation (its origins are dated back to the paper [6]) coupled with the nonlinear ordinary differential equations $(1.1)_4$ describing inelastic deformations (so-called constitutive equation).

In this work we assume that the porous media with the material density $\rho > 0$ lies within the subset $\Omega \subset \mathbb{R}^3$. The dynamical poroplasticity model can be written in the form

$$\begin{aligned} \rho u_{tt}(x,t) - \mathrm{div}_x T(x,t) + \alpha \nabla_x p(x,t) &= F(x,t), \\ c_0 p_t(x,t) - c\Delta_x p(x,t) + \alpha \mathrm{div}_x u_t(x,t) &= f(x,t), \\ T(x,t) &= \mathcal{D}\left(\varepsilon\left(u(x,t)\right) - \varepsilon^p(x,t)\right), \\ \varepsilon_t^p(x,t) &= A(T(x,t)), \end{aligned} \tag{1.1}$$

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http://dx.doi.org/10.1016/j.jmaa.2017.01.045

where $\varepsilon(u(x,t))$ means the symmetric part of the gradient of function $u(x,t)$ i.e.

$$\varepsilon(u(x,t)) = \frac{1}{2}\left(\nabla_x u(x,t) + \nabla_x^T u(x,t)\right).$$

The first equation $(1.1)_1$ is the balance of momentum coupled with the generalized Hook law (equation $(1.1)_3$), the second equation $(1.1)_2$ is a combination of the Darcy law and the mass conservation law for a fluid. For any fixed $T_e > 0$ we are interested in finding the following

- the displacement field $u : \Omega \times [0, T_e] \to \mathbb{R}^3$,
- the pore pressure of the fluid $p : \Omega \times [0, T_e] \to \mathbb{R}$,
- the inelastic deformation tensor $\varepsilon^p : \Omega \times [0, T_e] \to \mathcal{S}(3) = \mathbb{R}^{3\times 3}_{sym}$,
- the Cauchy stress tensor $T : \Omega \times [0, T_e] \to \mathcal{S}(3)$.

The given functions $F : \Omega \times [0, T_e] \to \mathbb{R}^3$, and $f : \Omega \times [0, T_e] \to \mathbb{R}$ describe a density of applied body forces and a force of fluid extraction or injection process, respectively. Moreover, $\mathcal{D} : \mathcal{S}(3) \to \mathcal{S}(3)$ is a linear, symmetric and positive-definite elasticity tensor which is assumed to be constant in time and space, $A : \mathcal{S}(3) \to \mathcal{S}(3)$ is a given inelastic constitutive function and ρ, α, c, c_0 are the positive material constants (for details see [28]).

Problem (1.1) will be considered with the mixed boundary conditions

$$\begin{aligned} u(x,t) &= g_D(x,t), \qquad && x \in \Gamma_D,\ t \geqslant 0,\\ (T(x,t) - \alpha p(x,t)\mathbb{I})n(x) &= g_N(x,t), && x \in \Gamma_N,\ t \geqslant 0,\\ p(x,t) &= g_P(x,t), && x \in \Gamma_P,\ t \geqslant 0,\\ c\frac{\partial p}{\partial n}(x,t) &= g_V(x,t), && x \in \Gamma_V,\ t \geqslant 0, \end{aligned} \tag{1.2}$$

where $n(x)$ is the outward pointing, unit normal vector at point $x \in \partial\Omega$ and the sets: Γ_D, Γ_N, Γ_P, Γ_V are open subsets of $\partial\Omega$ such that

- $\mathcal{H}_2(\Gamma_D) > 0$, $\mathcal{H}_2(\Gamma_P) > 0$, where $\mathcal{H}_2$ denotes the two-dimensional Hausdorff measure.
- $\partial\Omega = \overline{\Gamma}_D \cup \overline{\Gamma}_N = \overline{\Gamma}_P \cup \overline{\Gamma}_V, \quad \Gamma_D \cap \Gamma_N = \Gamma_P \cap \Gamma_V = \emptyset.$

We also need the following initial conditions

$$\begin{aligned} u(x,0) &= u_0(x), \qquad && x \in \Omega,\\ u_t(x,0) &= u_1(x), && x \in \Omega,\\ p(x,0) &= p_0(x), && x \in \Omega,\\ \varepsilon^p(x,0) &= \varepsilon^p_0(x), && x \in \Omega. \end{aligned} \tag{1.3}$$

In the paper we assume that Ω is an open, bounded and smooth subset of $\mathbb{R}^3$ and the inelastic constitutive function A is deviatoric, i.e.

$$A : \mathcal{S}(3) \to P\mathcal{S}(3), \qquad \text{where} \qquad PT = T - \frac{1}{3}\mathrm{Tr}(T)\mathbb{I}. \tag{1.4}$$

We also assume that A has the following form