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Generalized Zalcman conjecture for some classes of analytic functions ☆

V. Ravichandran, Shelly Verma *

Department of Mathematics, University of Delhi, Delhi-110 007, India

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A B S T R A C T

For functions $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ in various subclasses of normalized analytic functions, we consider the problem of estimating the generalized Zalcman coefficient functional $\phi(f, n, m; \lambda) := |\lambda a_n a_m - a_{n+m-1}|$. For all real parameters λ and $\beta < 1$, we provide the sharp upper bound of $\phi(f, n, m; \lambda)$ for functions f satisfying $\mathrm{Re} f'(z) > \beta$ and hence settle the open problem of estimating $\phi(f, n, m; \lambda)$ recently proposed by Agrawal and Sahoo (2016) [1]. For all real values of λ, the estimations of $\phi(f, n, m; \lambda)$ are provided for starlike and convex functions of order α ($\alpha < 1$) which are sharp for $\lambda \le 0$ or for certain positive values of λ. Moreover, for certain positive λ, the sharp estimation of $\phi(f, n, m; \lambda)$ is given when f is a typically real function or a univalent function with real coefficients or is in some subclasses of close-to-convex functions.

1. Introduction and preliminaries

Let $\mathcal{A}$ be the class of all normalized analytic functions of the form $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ defined on the open unit disc $\mathbb{D}$. The subclass of $\mathcal{A}$ consisting of univalent functions is denoted by $\mathcal{S}$. Let $\mathcal{S}_{\mathbb{R}}$ be the class of all functions in $\mathcal{S}$ with real coefficients. For $\alpha < 1$, we denote by $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$, the classes of functions $f \in \mathcal{A}$ satisfying $\mathrm{Re}\big(zf'(z)/f(z)\big) > \alpha$ and $\mathrm{Re}\big(1 + zf''(z)/f'(z)\big) > \alpha$ respectively. For $0 \le \alpha < 1$, these classes are subclasses of $\mathcal{S}$ and were first introduced by Robertson [22] in 1936. Later, for all $\alpha < 1$, these classes were considered in [23,4]. The classes $\mathcal{S}^* := \mathcal{S}^*(0)$ and $\mathcal{K} := \mathcal{K}(0)$ represent the classes of starlike and convex functions respectively. We denote the closed convex hulls of $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ by $H\mathcal{S}^*(\alpha)$ and $H\mathcal{K}(\alpha)$ respectively. The class of typically real functions, denoted by T, consists of all functions in $\mathcal{A}$ which have real values on the real axis and non-real values elsewhere. Denote by $\mathcal{P}$, the class of all analytic functions $p(z) = 1 + c_1z + c_2z^2 + \cdots$ defined on $\mathbb{D}$ such that $\mathrm{Re} p(z) > 0$. The class $\mathcal{P}_{\mathbb{R}}$ consists of all functions in $\mathcal{P}$ with real coefficients.

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* Corresponding author.
E-mail addresses: vravi68@gmail.com, vravi@maths.du.ac.in (V. Ravichandran), jmdsv.maths@gmail.com (S. Verma).

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In 1916, Bieberbach conjectured the inequality $|a_n| \leq n$ for $f \in \mathcal{S}$. Since then, several attempts were made to prove the Bieberbach conjecture which was finally proved by de Branges in 1985. In 1960, as an approach to prove the Bieberbach conjecture, Lawrence Zalcman conjectured that $|a_n^2 - a_{2n-1}| \leq (n-1)^2$ $(n \geq 2)$ for $f \in \mathcal{S}$. This led to several works related to Zalcman conjecture and its generalized version $|\lambda a_n^2 - a_{2n-1}| \leq \lambda n^2 - 2n + 1$ $(\lambda \geq 0)$ for various subclasses of $\mathcal{S}$ [18,20,6,16,5,14] but the Zalcman conjecture remained open for many years for the class $\mathcal{S}$. However, for $n \leq 6$, Krushkal [11] proved the conjecture for the class $\mathcal{S}$ by using holomorphic homotopy of univalent functions and with the similar geometric idea, he has recently proved it for all $n \geq 2$ in his unpublished work [12].

In 1999, Ma [19] proposed a generalized Zalcman conjecture for $f \in \mathcal{S}$ that

$$|a_n a_m - a_{n+m-1}| \leq (n-1)(m-1) \quad (n \geq 2, m \geq 2)$$

which is still an open problem, however he proved it for the classes $\mathcal{S}^*$ and $\mathcal{S}_{\mathbb{R}}$. For $\lambda \in \mathbb{R}$, let $\phi(f,n,m;\lambda) := |\lambda a_n a_m - a_{n+m-1}|$ denote the generalized Zalcman coefficient functional over $\mathcal{A}$. For $\beta < 1$, the class $\mathcal{C}(\beta)$ of close-to-convex functions of order β consists of $f \in \mathcal{A}$ such that $\operatorname{Re}\big(zf'(z)/\big(e^{i\theta}g(z)\big)\big) > \beta$ for some $g \in \mathcal{S}^*$ and $\theta \in \mathbb{R}$. For $0 \leq \beta < 1$, the class $\mathcal{C}(\beta)$ is a subclass of $\mathcal{S}$ and was considered in [17] in a more general form. The class of close-to-convex functions is denoted by $\mathcal{C} := \mathcal{C}(0)$, for details, see [8]. Let $\mathcal{F}_1(\beta)$ and $\mathcal{F}_2(\beta)$ be the subclasses of $\mathcal{C}(\beta)$ $(\beta < 1)$ corresponding to $\theta = 0$ and the starlike functions $g(z) = z/(1-z)$ and $g(z) = z/(1-z^2)$ respectively. For $\beta < 1$, let $\mathcal{R}(\beta)$ denote the class of functions $f \in \mathcal{A}$ satisfying $\operatorname{Re} f'(z) > \beta$. For $0 \leq \beta < 1$, $\mathcal{R}(\beta)$ is a subclass of $\mathcal{S}$ and was first introduced in [9]. Here, we are interested in $\mathcal{R}(\beta)$ for all values of β $(\beta < 1)$. Recently, for some positive values of λ and $0 \leq \beta < 1$, Agrawal and Sahoo [1] gave the sharp estimation of $\phi(f,n,m;\lambda)$ for the classes $\mathcal{R}(\beta)$ and $H\mathcal{K}$.

In this paper, for all real values of λ, we give the sharp estimation of $\phi(f,n,m;\lambda)$ for $f \in \mathcal{R}(\beta)$ $(\beta < 1)$. Also, for $f \in \mathcal{S}^*(\alpha)$ and $f \in \mathcal{K}(\alpha)$ $(\alpha < 1)$, the estimations of $\phi(f,n,m;\lambda)$ are given for all real values of λ which are sharp when $\lambda \leq 0$ or when λ is taking certain positive values. Moreover, for certain positive values of λ, the sharp estimations of $\phi(f,n,m;\lambda)$ are provided for the classes T, $\mathcal{S}_{\mathbb{R}}$, $\mathcal{F}_1(\beta)$ and $\mathcal{F}_2(\beta)$ $(\beta < 1)$.

We prove our results either by applying the well-known estimation of $|\lambda c_n c_m - c_{n+m}|$ for $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \in \mathcal{P}$ or by applying some characterization of functions in the class $\mathcal{P}$ and that of typically real functions in terms of some positive semi-definite Hermitian form, see [13,21]. Earlier, such characterization of functions with positive real part in terms of some positive semi-definite Hermitian form [13] was used in [3,2,21]. It should be pointed out that in the literature, for various subclasses of $\mathcal{S}$ which are invariant under rotations, the estimation of $\phi(f,n,n;\lambda)$ is usually obtained by using the fact that the expression $\phi(f,n,n;\lambda)$ is invariant under rotations and by an application of the Cauchy–Schwarz inequality which requires λ to be non-negative. However, we are able to give the sharp estimation of $\phi(f,n,m;\lambda)$ for various subclasses of $\mathcal{A}$ when $\lambda \leq 0$. Moreover, for certain positive λ, our technique is giving the estimation of $\phi(f,n,m;\lambda)$ when f is in some subclasses of $\mathcal{A}$ which are not necessarily invariant under rotations. We need the following lemmas to prove our results.

Lemma 1.1. *[21, Lemma 2.3, p. 507] If $p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k \in \mathcal{P}$, then for all $n, m \in \mathbb{N}$,*

$$|\mu c_n c_m - c_{n+m}| \leq \begin{cases} 2, & 0 \leq \mu \leq 1; \\ 2|2\mu - 1|, & \text{elsewhere.} \end{cases}$$

The result is sharp.