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The index of weighted singular integral operators with shifts and slowly oscillating data

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ABSTRACT

Let α and β be orientation-preserving diffeomorphism (shifts) of $\mathbb{R}_+ = (0, \infty)$ onto itself with the only fixed points 0 and ∞. We establish a Fredholm criterion and calculate the index of the weighted singular integral operator with shifts

$$(aI - bU_\alpha)P_\gamma^+ + (cI - dU_\beta)P_\gamma^-,$$

acting on the space $L^p(\mathbb{R}_+)$, where $P_\gamma^\pm = (I \pm S_\gamma)/2$ are the operators associated to the weighted Cauchy singular integral operator S_γ given by

$$(S_\gamma f)(t) = \frac{1}{\pi i} \int_{\mathbb{R}_+} \left(\frac{t}{\tau}\right)^\gamma \frac{f(\tau)}{\tau - t} d\tau$$

with $\gamma \in \mathbb{C}$ satisfying $0 < 1/p + \Re\gamma < 1$, and U_α, U_β are the isometric shift operators given by

$$U_\alpha f = (\alpha')^{1/p}(f \circ \alpha), \quad U_\beta f = (\beta')^{1/p}(f \circ \beta),$$

under the assumptions that the coefficients a, b, c, d and the derivatives α', β' of the shifts are bounded and continuous on $\mathbb{R}_+$ and admit discontinuities of slowly oscillating type at 0 and ∞.

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1. Introduction

Let $\mathcal{B}(X)$ be the Banach algebra of all bounded linear operators acting on a Banach space X and let $\mathcal{K}(X)$ be the ideal of all compact operators in $\mathcal{B}(X)$. An operator $A \in \mathcal{B}(X)$ is called *Fredholm* if its image is closed and the spaces $\ker A$ and $\ker A^*$ are finite-dimensional. In that case the number

$$\operatorname{Ind} A := \dim\ker A - \dim\ker A^*$$

is referred to as the *index* of A (see, e.g., [5, Chap. 4]). For $A, B \in \mathcal{B}(X)$, we will write $A \simeq B$ if $A - B \in \mathcal{K}(X)$.

Following Sarason [26, p. 820], a bounded continuous function f on $\mathbb{R}_+ = (0, \infty)$ is called slowly oscillating (at 0 and ∞) if

$$\lim_{r\to s} \sup_{t,\tau\in[r,2r]} |f(t) - f(\tau)| = 0 \quad \text{for} \quad s \in \{0, \infty\}.$$

The set $SO(\mathbb{R}_+)$ of all slowly oscillating functions forms a C^*-algebra. This algebra properly contains $C(\overline{\mathbb{R}}_+)$, the C^*-algebra of all continuous functions on $\overline{\mathbb{R}}_+ := [0, +\infty]$.

Suppose α is an orientation-preserving diffeomorphism of $\mathbb{R}_+$ onto itself, which has only two fixed points 0 and ∞. We say that α is a slowly oscillating shift if $\log \alpha'$ is bounded and $\alpha' \in SO(\mathbb{R}_+)$. The set of all slowly oscillating shifts is denoted by $SOS(\mathbb{R}_+)$. By [7, Lemma 2.2], an orientation-preserving diffeomorphism $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to $SOS(\mathbb{R}_+)$ if and only if $\alpha(t) = te^{\omega(t)}$, $t \in \mathbb{R}_+$, for some real-valued function $\omega \in SO(\mathbb{R}_+) \cap C^1(\mathbb{R}_+)$ such that $\psi(t) := t\omega'(t)$ also belongs to $SO(\mathbb{R}_+)$ and $\inf_{t\in\mathbb{R}_+} \big(1 + t\omega'(t)\big) > 0$. The real-valued slowly oscillating function

$$\omega(t) := \log[\alpha(t)/t], \quad t \in \mathbb{R}_+,$$

is called the exponent function of $\alpha \in SOS(\mathbb{R}_+)$.

Through the paper, we will suppose that $1 < p < \infty$. It is easily seen that if $\alpha \in SOS(\mathbb{R}_+)$, then the weighted shift operator defined by

$$U_\alpha f := (\alpha')^{1/p}(f \circ \alpha)$$

is an isometric isomorphism of the Lebesgue space $L^p(\mathbb{R}_+)$ onto itself. It is clear that $U_\alpha^{-1} = U_{\alpha_{-1}}$. Let $a, b \in SO(\mathbb{R}_+)$. We say that a dominates b and write $a \gg b$ if

$$\inf_{t\in\mathbb{R}_+} |a(t)| > 0, \quad \liminf_{t\to 0}(|a(t)| - |b(t)|) > 0, \quad \liminf_{t\to\infty}(|a(t)| - |b(t)|) > 0.$$

Theorem 1.1 *([13, Theorem 1.1]). Suppose $a, b \in SO(\mathbb{R}_+)$ and $\alpha \in SOS(\mathbb{R}_+)$. The binomial functional operator $aI - bU_\alpha$ is invertible on the Lebesgue space $L^p(\mathbb{R}_+)$ if and only if either $a \gg b$ or $b \gg a$.*

(a) *If $a \gg b$, then* $(aI - bU_\alpha)^{-1} = \sum_{n=0}^{\infty} (a^{-1}bU_\alpha)^n a^{-1}I.$

(b) *If $b \gg a$, then* $(aI - bU_\alpha)^{-1} = -U_\alpha^{-1} \sum_{n=0}^{\infty} (b^{-1}aU_\alpha^{-1})^n b^{-1}I.$

Let $\Re\gamma$ and $\Im\gamma$ denote the real and imaginary part of $\gamma \in \mathbb{C}$, respectively. As usual, $\overline{\gamma} = \Re\gamma - i\Im\gamma$ denotes the complex conjugate of γ. If $\gamma \in \mathbb{C}$ satisfies

$$0 < 1/p + \Re\gamma < 1, \tag{1.1}$$

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