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On the geometry of random convex sets between polytopes and zonotopes

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ABSTRACT

In this work we study a class of random convex sets that "interpolate" between polytopes and zonotopes. These sets arise from considering a q^{th}-moment ($q \geq 1$) of an average of order statistics of 1-dimensional marginals of a sequence of $N \geq n$ independent random vectors in $\mathbb{R}^n$. We consider the random model of isotropic log-concave distributions as well as the uniform distribution on an ℓ_p^n-sphere ($1 \leq p < \infty$) with respect to the cone probability measure, and study the geometry of these sets in terms of the support function and mean width. We provide asymptotic formulas for the expectation of these geometric functionals which are sharp up to absolute constants. Our model includes and generalizes the standard one for random polytopes.

1. Introduction and main results

1.1. General introduction

A random polytope in $\mathbb{R}^n$ is the convex hull of N points chosen randomly according to a given law. In fact, several other models to define random polytopes exist, but this model is arguably the most natural, best known and most studied one. It was more than 150 years ago that J. J. Sylvester initiated their study when he posed a problem in The Educational Times in 1864 [44]. In it, he asked for the probability that four points chosen uniformly at random in an indefinite plane have a convex hull which is a four-sided polygon. Within a year it was understood that the question was ill-posed and Sylvester modified the question, asking for the probability that four points chosen independently and uniformly at random from a convex set K in the plane are in convex position. This problem became known as the famous "four-point problem" and was the starting point of extensive research (see also [4] and the references therein).

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It were A. Rényi and R. Sulanke who later, in their seminal papers [39–41], focused on the asymptotic of the expected volume of a random polytope as the number of points N tends to infinity. Since then and especially in the last decades, random polytopes found increasing interest. This is to a large extent due to their emergence in various branches of mathematics and their broad spectrum of applications. Among others, random polytopes appear in approximation theory [31,5], random matrix theory [26] or in other disciplines such as statistics, information theory, signal processing, medical imaging or digital communications (see [12] and the references therein), just to mention a few. Because of their "pathologically" bad behavior, they are also a major source for counterexamples, as can be seen, for instance, in [14] or [27]. Some of the important quantities studied in order to understand their geometric structure are expectations, variances, and distributions of functionals associated to the random polytope, for instance, the volume, the number of vertices, intrinsic volumes, mean outer radii and, in particular, the mean width.

Obviously, the behavior of these geometric functionals depends on the underlying model of randomness. There are two such models that have drawn a particularly lot of attention and have been studied extensively. One situation is the case in which the random vectors generating the polytope are Gaussian and results in this direction can be found, for instance, in [14,45,29,22,26,23] and the references given therein. The other one is the case when the points that span the polytope are chosen uniformly at random inside a convex body K. Here, we may refer the reader to [10,11,2,1] and again the references given there. Typically, K is considered to be isotropic and the geometry of the random polytope relates to the isotropic constant of K. These two models are particular situations of the case when the random vectors are distributed according to an isotropic log-concave probability, which is the general framework in which they are studied.

In the work [19], extending the previous works [15,17,16], Y. Gordon, A. E. Litvak, C. Schütt and E. Werner studied the geometry of the unit balls and their polars of the norm given by

$$\|x\|_{\ell,q} = \left(\sum_{k=1}^{\ell} \underset{1\le i\le N}{\text{k-max}} |\langle x, a_i\rangle|^q \right)^{1/q}, \qquad x \in \mathbb{R}^n,$$

where $1 \le q \le \infty$, $\{a_i\}_{i=1}^N$ is a fixed sequence of vectors spanning $\mathbb{R}^n$, $1 \le \ell \le N$, and $\text{k-max}_{1\le i\le N} |\langle x, a_i\rangle|$ is the k^{th} largest number in the set $\{|\langle x, a_i\rangle|\}_{i=1}^N$. As different choices of the involved parameters show, this class of convex bodies is quite rich, which also explains the interest in those spaces. To be more precise, when we choose $\ell = 1$, then the polar body of this unit ball is the symmetric convex hull of the vectors $a_1, \dots, a_N$. On the other hand, if we let $\ell = N$, then the polar of the unit ball of this norm is just a linear transformation of a projection of the unit ball of $\ell_{q^*}^N$ onto an n-dimensional subspace, where q^* is the conjugate of q. In particular, choosing $q = 1$ and $\ell = N$, the polar body of the unit ball of $\|\cdot\|_{\ell,q}$ is a zonotope. For $q = 1$, the polar of the unit ball is a linear image of a projection of $\left(\ell\, \mathbb{B}_1^N\right) \cap \mathbb{B}_\infty^N$ (see Lemma 5.1 in [19]).

Here, we introduce a probabilistic variant of this by considering the vectors $a_1, \dots, a_N \in \mathbb{R}^n$ not to be fixed, but chosen independently at random according to a given probability law on $\mathbb{R}^n$ (details are given below). This is, in fact, quite interesting and natural, because a rich family of random convex sets arises that includes the important class of random polytopes, but extends beyond that classical and well understood setting. In this new model, the definition of the random convex sets takes more order statistics of 1-dimensional marginals and higher moments into account. It therefore should capture more information about the geometry and distribution of mass. In this work, we study how sensitive this information is in the number of order statistics and moments considered and initiate the study of this new and more general class of random convex sets, restricting ourselves to the expectation of the mean width for now. To be more precise, we will study the geometry of this family of random convex bodies for several models of randomness and the dependence of their geometric parameters on the space dimension n, the number N of vectors generating them, the number ℓ of order statistics considered and their dependence on the moment q. We will compute, up to absolute constants, the expected value of the mean width of the polar bodies of the unit balls of $\|\cdot\|_{\ell,q}$ when the independent random vectors $a_1, \dots, a_N$ are distributed

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