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C^*-algebras associated to Boolean dynamical systems

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ABSTRACT

The goal of these notes is to present the C^*-algebra $C^*(\mathcal{B}, \mathcal{L}, \theta)$ of a Boolean dynamical system $(\mathcal{B}, \mathcal{L}, \theta)$, that generalizes the C^*-algebra associated to labelled graphs introduced by Bates and Pask, and to determine its simplicity, its gauge invariant ideals, as well as compute its K-Theory.

1. Introduction

In 1980 Cuntz and Krieger [11] associated a C^*-algebra $\mathcal{O}_A$ to a shift of finite type with transition matrix A. Various authors – including Bates, Fowler, Kumjian, Laca, Pask and Raeburn – extended the original construction to more general subshifts associated with directed graphs, giving origin to the graph C^*-algebra $C^*(E)$ of a directed graph E (see e.g. [20,28]). Using a different approach, Exel and Laca [17] generalize Cuntz–Krieger algebras, by associating a C^*-algebra to an infinite matrix which 0 and 1 entries. Later, Tomforde [33] introduced the class of ultragraph algebras in order to unify Exel–Laca algebras and graph C^*-algebras. Also, motivated by Cuntz–Krieger construction, Matsumoto [31] introduced a C^*-algebra associated with a general two-sided subshift over a finite alphabet. Later, the first named author [8] extended Matsumoto's construction, by constructing the C^*-algebra $\mathcal{O}_\Lambda$ associated with a general one-sided subshift Λ over a finite alphabet.

One of the underlying ideas of associating a C^*-algebra to a dynamical system comes from the Franks classification of irreducible shifts of finite type up to flow equivalence [21]. This classification uses the Bowen–

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Franks group of the shift space, that turns out to be the K_0-group of the associated Cuntz–Krieger algebra [11]. Therefore, the idea was to study the connection between classification of shift spaces and classification of C^*-algebras. Following this point of view, the recent results of Matsumoto and Matui [32] characterize continuous orbit equivalence of shifts of finite type by using K-theoretical invariants of the associated C^*-algebra. It is natural to try to extend the scope of this strategy to classify shift space over a finite alphabet. By adapting the left-Krieger cover construction given in [27], any shift space over a finite alphabet may be presented by a left-resolving labelled graph. Thus, in the same spirit of the previous constructions, labelled graph algebras, introduced by Bates and Pask in [1], provided a method for associating a C^*-algebra to a shift space over a finite alphabet. The class of labelled graph C^*-algebras contains, in particular, all the above mentioned classes of C^*-algebra. Properties like simplicity, ideal structure and purely infinity was studied in [2,24] and the computation of the K-theory was achieved in [3].

The original goal of the present paper was to continue the study of the labelled graph C^*-algebras, by characterizing them as 0-dimensional topological graphs [25]. However, the topological graph E associated to the data of the labelled graph is just a realization of a Boolean algebra of a family of subsets of vertices of E, plus some partial actions given by the arrows of E. Thus, we adapt the labelled graph C^*-algebra construction, as well as our topological graph characterization, to the context of a C^*-algebra associated to a general family of partial actions over a fixed Boolean algebra (we call it a Boolean dynamical system). This class of C^*-algebras, that we call Boolean Cuntz–Krieger algebras associated with a Boolean dynamical systems, includes labelled graph C^*-algebras, homeomorphism C^*-algebras over 0-dimensional compact spaces, and graph C^*-algebras, among others. Essentially, it is not a new class of C^*-algebras, since they are (0-dimensional) algebras over topological graphs, a class deeply studied by Katsura [25,26]. However, the advantage of our approach is that we can skip to deal with the topology of the graph, and instead can concentrate only in combinatorial properties of actions over a Boolean algebra. In particular, we can use a different picture when studying C^*-algebras associated to combinatorial objects, by using groupoid C^*-algebras. This is a classical approach, used by Kumjian, Pask, Raeburn and Renault [28] when studying graph C^*-algebras. This approach attained a new level of efficiency when Exel [13] developed a huge machinery that helps to represent any "combinatorial" C^*-algebra as a full groupoid C^*-algebra. The strategy is to associate to the C^*-algebra an $*$-inverse semigroup (see e.g. [29]) and a "tight" representation (i.e. a representations preserving additive identities on pairwise orthogonal idempotents). When this is possible, there is a standard way of producing a étale, second countable topological groupoid which full C^*-algebra is isomorphic to the original C^*-algebra under consideration. In the case of Boolean Cuntz–Krieger algebras associated to Boolean dynamical system this strategy works, and so we can use all the machinery developed by Exel [13,14] for analyze the structure of the algebras under study. Recent examples of application of such an strategy are [18,19].

The contents of this paper can be summarized as follows: In Section 2 we recall some Boolean algebra theory. In particular, we summarize some well-known results about the topology of the space of characters (the Stone's spectrum) of a Boolean algebra. In Section 3 we define Boolean dynamical systems, that are families of partial actions on a Boolean algebra, and their representations in a C^*-algebra; the C^*-algebra associated to the universal representation will be the Boolean Cuntz–Krieger algebra. We state the existence of a universal representation and the gauge uniqueness theorem, that will be proved later. In Section 4 we recall the definition of Katsura's topological graph. When E is a 0-dimensional space, i.e. both the vertex and edge spaces are 0-dimensional, we construct a Boolean dynamical system that can be represented in the associated topological graph C^*-algebra $\mathcal{O}(E)$. In Section 5 we focus on finding a universal representation of a given Boolean dynamical system. This is achieved by constructing a compactly supported 0-dimensional topological graph with the data of the Boolean dynamical system, and defining a representation of the Boolean dynamical system in the topological graph C^*-algebra. We conclude proving that the Boolean Cuntz–Krieger algebras are isomorphic to a 0-dimensional topological graph C^*-algebra, and using this characterization to compute its K-Theory. In Sections 6, 7 and 8 we apply Exel's machinery to