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Relations between Schramm spaces and generalized Wiener classes

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ABSTRACT

We give necessary and sufficient conditions for the embeddings $\Lambda\mathrm{BV}^{(p)} \subseteq \Gamma\mathrm{BV}^{(q_n\uparrow q)}$ and $\Phi\mathrm{BV} \subseteq \mathrm{BV}^{(q_n\uparrow q)}$. As a consequence, a number of results in the literature, including a fundamental theorem of Perlman and Waterman, are simultaneously extended.

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1. Introduction and main results

Let $\Lambda = \{\lambda_j\}_{j=1}^{\infty}$ be a nondecreasing sequence of positive numbers such that $\sum_{j=1}^{\infty} \frac{1}{\lambda_j} = \infty$. Following [1], we call Λ a Waterman sequence. Let $\Phi = \{\phi_j\}_{j=1}^{\infty}$ be a sequence of increasing convex functions on $[0,\infty)$ with $\phi_j(0) = 0$. We say that Φ is a Schramm sequence if $0 < \phi_{j+1}(x) \le \phi_j(x)$ for all j and $\sum_{j=1}^{\infty} \phi_j(x) = \infty$ for all $x > 0$. This terminology is used throughout.

We begin by recalling two generalizations of the concept of bounded variation which are central to our work.

Definition 1.1. A real-valued function f on $[a,b]$ is said to be of Φ-bounded variation if

$$V_\Phi(f) = V_\Phi(f;[a,b]) = \sup \sum_{j=1}^{n} \phi_j(|f(I_j)|) < \infty,$$

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where the supremum is taken over all finite collections $\{I_j\}_{j=1}^n$ of nonoverlapping subintervals of $[a,b]$ and $f(I_j) = f(\sup I_j) - f(\inf I_j)$. We denote by ΦBV the linear space of all functions f such that cf is of Φ-bounded variation for some $c > 0$.

If for every $f \in \Phi\mathrm{BV}$, we define

$$\|f\| := |f(a)| + \inf\{c > 0 : V_\Phi(f/c) \leq 1\},$$

then it is easily seen that $\|\cdot\|$ is a norm, and ΦBV endowed with this norm turns into a Banach space. The space ΦBV is introduced in Schramm's paper [15]. For more information about ΦBV, the reader is referred to [1].

If ϕ is a strictly increasing convex function on $[0,\infty)$ with $\phi(0) = 0$, and if $\Lambda = \{\lambda_j\}_{j=1}^\infty$ is a Waterman sequence, by taking $\phi_j(x) = \phi(x)/\lambda_j$ for all j, we get the class $\phi\Lambda\mathrm{BV}$ of functions of $\phi\Lambda$-bounded variation. This class was introduced by Schramm and Waterman in [16] (see also [17] and [11]). More specifically, if $\phi(x) = x^p$ $(p \geq 1)$, we get the Waterman–Shiba class $\Lambda\mathrm{BV}^{(p)}$, which was introduced by Shiba in [18]. When $p = 1$, we obtain the well-known Waterman class ΛBV.

In the case $\lambda_j = 1$ for all j, we obtain the class $\phi\mathrm{BV}$ of functions of ϕ-bounded variation introduced by Young [26]. More specifically, when $\phi(x) = x^p$ $(p \geq 1)$, we obtain the Wiener class BV_p (see [24]), and taking $p = 1$, we have the well-known Jordan class BV.

Remark 1.2. One can easily observe that functions of Φ-bounded variation are bounded and can only have simple discontinuities (countably many of them, indeed). The class ΦBV has many applications in Fourier analysis as well as in treating topics such as convergence, summability, etc. (see [24,26,21–23,12,15]).

Definition 1.3. Let $\{q_n\}_{n=1}^\infty$ and $\{\delta_n\}_{n=1}^\infty$ be sequences of positive real numbers such that $1 \leq q_n \uparrow q \leq \infty$ and $2 \leq \delta_n \uparrow \infty$. A real-valued function f on $[a,b]$ is said to be of q_n-Λ-bounded variation if

$$V_\Lambda(f) = V_\Lambda(f; q_n \uparrow q; \delta) := \sup_{n\geq 1} \sup_{\{I_j\}} \Big(\sum_{j=1}^{s} \frac{|f(I_j)|^{q_n}}{\lambda_j} \Big)^{\frac{1}{q_n}} < \infty,$$

where the $\{I_j\}_{j=1}^s$ are collections of nonoverlapping subintervals of $[a,b]$ such that $\inf_j |I_j| \geq \frac{b-a}{\delta_n}$. The class of functions of q_n-Λ-bounded variation is denoted by $\Lambda\mathrm{BV}^{(q_n\uparrow q)}$ $(= \Lambda\mathrm{BV}_\delta^{(q_n\uparrow q)})$. In the sequel, we suppose that $[a,b] = [0,1]$.

The class $\Lambda\mathrm{BV}^{(q_n\uparrow q)}$ was introduced by Vyas in [19]. When $\lambda_j = 1$ for all j and $\delta_n = 2^n$ for all n, we get the class $\mathrm{BV}^{(q_n\uparrow q)}$—introduced by Kita and Yoneda (see [9])—which in turn recedes to the Wiener class BV_q, when $q_n = q$ for all n.

A natural and important problem is to determine relations between the above-mentioned classes; see [21,12,4,9,6,13,8,5] for some results in this direction. In particular, Perlman and Waterman found the fundamental characterization of embeddings between ΛBV classes in [12]. Ge and Wang characterized the embeddings $\Lambda\mathrm{BV} \subseteq \phi\mathrm{BV}$ and $\phi\mathrm{BV} \subseteq \Lambda\mathrm{BV}$ (see [5]). It was shown by Kita and Yoneda in [9] that the embedding $\mathrm{BV}_p \subseteq \mathrm{BV}^{(p_n\uparrow\infty)}$ is both automatic and strict for all $1 \leq p < \infty$. Furthermore, Goginava characterized the embedding $\Lambda\mathrm{BV} \subseteq \mathrm{BV}^{(q_n\uparrow\infty)}$ in [6], and a characterization of the embedding $\Lambda\mathrm{BV}^{(p)} \subseteq \mathrm{BV}^{(q_n\uparrow q)}$ $(1 \leq q \leq \infty)$ was given by Hormozi, Prus-Wiśniowski and Rosengren in [8]. In this paper, we investigate the embeddings $\Lambda\mathrm{BV}^{(p)} \subseteq \Gamma\mathrm{BV}^{(q_n\uparrow q)}$ and $\Phi\mathrm{BV} \subseteq \mathrm{BV}^{(q_n\uparrow q)}$ $(1 \leq q \leq \infty)$. The problem as to when the reverse embeddings hold is also considered, which turns out to have a simple answer (see Remark 1.10(ii) below).

Throughout this paper, the letters Λ and Γ are reserved for a typical Waterman sequence. We associate to Λ a function which we still denote by Λ and define it as $\Lambda(r) := \sum_{j=1}^{[r]} \frac{1}{\lambda_j}$ for $r \geq 1$. The function $\Lambda(r)$ is clearly nondecreasing and $\Lambda(r) \to \infty$ as $r \to \infty$. Our first main result reads as follows.

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