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Classes of Fourier–Feynman transforms on Wiener space

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A R T I C L E I N F O

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In this article we examine several classes of analytic Fourier–Feynman transforms on Wiener space. The classes investigated in this article form commutative monoids (and hence semigroups).

1. Introduction

The main purpose of this article is to classify some of the classes of the analytic Fourier–Feynman transforms (FFT) using an algebraic viewpoint. The analytic FFT is a well-known transform defined on infinite dimensional linear spaces.

Let $C_0[0,T]$ denote one-parameter Wiener space, that is, the space of all real-valued continuous functions x on $[0,T]$ with $x(0) = 0$. Let $\mathcal{M}$ denote the class of all Wiener measurable subsets of $C_0[0,T]$ and let $\mathfrak{m}$ denote Wiener measure which is a Gaussian measure on $C_0[0,T]$ with mean zero and covariance function $r(s,t) = \min\{s,t\}$. Then, as is well known, $(C_0[0,T], \mathcal{M}, \mathfrak{m})$ is a complete measure space. The concept of the analytic FFT of functionals on the Wiener space $C_0[0,T]$, introduced by Brue [1], has been developed in the literature. For instance, see [3,9,11,12]. This transform and its properties are similar in many respects to the ordinary Fourier transform of functions on Euclidean space. For an elementary introduction of the analytic FFT, see [20] and the references cited therein.

To explain what this transform is in its original context, let $C_0'[0,T]$ be the class of absolutely continuous functions x from $[0,T]$ to $\mathbb{R}$ for which $x(0) = 0$ and with $Dx \equiv dx/dt \in L_2[0,T]$, and let $\mathcal{D}$ be the non-existent Lebesgue measure on $C_0'[0,T]$. It is known that the space $C_0'[0,T]$ is an infinite dimensional separable Hilbert space. In the heuristic setting, the FFT of a functional F on H is defined by

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$$T_q(F)(y) = \exp\left\{\frac{iq}{2}\|y\|^2_{C_0'}\right\}\int\limits_{C_0'} \exp\{-iq(x,y)_{C_0'}\} \times F(x)\frac{1}{Z_q}\exp\left\{\frac{iq}{2}\|x\|^2_{C_0'}\right\}\mathcal{D}(x) \tag{1.1}$$

where $(\cdot,\cdot)_{C_0'}$ denotes the inner product given by $(x_1,x_2)_{C_0'} = \int_0^T Dx_1(t)Dx_2(t)dt$ (and hence $\|x\|^2_{C_0'}$ means the energy of the particle with the trajectory x), $\mathcal{D}$ is the heuristic version of Lebesgue measure, q is a nonzero real number, and Z_q is taken to be a normalization constant for which $\frac{1}{Z_q}\exp\{\frac{iq}{2}\|x\|^2_{C_0'}\}\mathcal{D}(x)$ is a probability measure on $C_0'[0,T]$. If we set $y = 0$ in (1.1), then equation (1.1) can be interpreted as the Feynman path integral. In [8], Feynman suggested $\mathcal{D}$ as a Lebesgue measure (namely, a translation invariant measure). As is widely known, there is not a true measure $\mathcal{D}$ on infinite dimensional spaces, and Z_q is, in fact, infinite. Even if this is a very heuristic description because of the formal observation above, one can see that 'Fourier' refers to the $\exp\{-iq(x,y)_{C_0'}\}$ term while 'Feynman' refers to the $\exp\{\frac{iq}{2}\|x\|^2_{C_0'}\}$ term in the integrand.

In order to furnish a rigorous definition of the FFT, let $(C_0'[0,T], C_0[0,T], \mathfrak{m})$ be the abstract Wiener space with $C_0'[0,T] \overset{i}{\hookrightarrow} C_0[0,T]$, where the natural inclusion i has a dense image under the supremum norm on $C_0[0,T]$, see [14]. For each $\lambda > 0$, let us use the usual informal expression for Wiener measure with variance λ^{-1} given by

$$d\mathfrak{m}_\lambda(x) = \frac{1}{Z_\lambda}\exp\left\{-\frac{\lambda}{2}\|x\|^2_{C_0'}\right\}\mathcal{D}(x).$$

Then a heuristic calculation shows that for $y \in C_0'[0,T]$,

$$\begin{aligned}
&\exp\left\{-\frac{\lambda}{2}\|y\|^2_{C_0'}\right\}\int\limits_{C_0'[0,T]} F(x)\exp\{\lambda(x,y)_{C_0'}\}d\mathfrak{m}_\lambda(x)\\
&= \exp\left\{-\frac{\lambda}{2}\|y\|^2_{C_0'}\right\}\int\limits_{C_0'[0,T]} F(x)\exp\{\lambda(x,y)_{C_0'}\}\frac{1}{Z_\lambda}\exp\left\{-\frac{\lambda}{2}\|x\|^2_{C_0'}\right\}\mathcal{D}(x)\\
&= \int\limits_{C_0'[0,T]} F(x)\frac{1}{Z_\lambda}\exp\left\{-\frac{1}{2}\left\|\sqrt{\lambda}(x-y)\right\|^2_{C_0'}\right\}\mathcal{D}(x)\\
&= \int\limits_{C_0'[0,T]} F(x+y)\frac{1}{Z_\lambda}\exp\left\{-\frac{1}{2}\left\|\sqrt{\lambda}x\right\|^2_{C_0'}\right\}\mathcal{D}(x)\\
&= \int\limits_{C_0'[0,T]} F(\lambda^{-1/2}x+y)\frac{1}{Z_1}\exp\left\{-\frac{1}{2}\|x\|^2_{C_0'}\right\}\mathcal{D}(x)\\
&= \int\limits_{C_0'[0,T]} F(\lambda^{-1/2}x+y)d\mathfrak{m}(x).
\end{aligned}$$

Thus we should expect that the FFT of F on $C_0[0,T]$ is given by

$$T_q(F)(y) = \lim_{\lambda\to -iq}\int\limits_{C_0[0,T]} F(\lambda^{-1/2}x+y)d\mathfrak{m}(x), \tag{1.2}$$

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