Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On a general class of optimal order multipoint methods for solving nonlinear equations

Janak Raj Sharma [a,*], Ioannis K. Argyros [b], Deepak Kumar [a]

[a] Department of Mathematics, Sant Longowal Institute of Engineering and Technology, Longowal-148106, Punjab, India

[b] Department of Mathematics Sciences, Cameron University, Lawton, OK 73505, USA

ARTICLE INFO

Article history:
Received 17 June 2016
Available online xxxx
Submitted by R.G. Durán

Keywords:
Nonlinear equations
Multipoint methods
Rational Hermite interpolation
Optimal convergence

ABSTRACT

We develop a class of n-point iterative methods with optimal 2^n order of convergence for solving nonlinear equations. Newton's second order and Ostrowski's fourth order methods are special cases corresponding to $n = 1$ and $n = 2$. Eighth and sixteenth order methods that correspond to $n = 3$ and $n = 4$ of the class are special cases of the eighth and sixteenth order methods proposed by Sharma et al. [25]. The methodology is based on employing the previously obtained $(n - 1)$-step scheme and modifying the n-th step by using rational Hermite interpolation. Unlike that of existing higher order techniques the proposed technique is attractive since it leads to a simple implementation. Local convergence analysis is provided to show that the iterations are locally well defined and convergent. Theoretical results are verified through numerical experimentations. The performance is also compared with already established methods in literature. It is observed that new algorithms are more accurate than existing counterparts and very effective in high precision computations.

1. Introduction

Multipoint iterative methods for solving nonlinear equation $f(x) = 0$, were initially studied in Ostrowski's book [19] and then they appeared extensively in Traub's book [29] and in recently published book by Petković et al. [21]. These methods are of great practical importance since they overcome the theoretical limits of one-point iterative methods regarding the computational order and efficiency. The multipoint methods were mainly introduced with the objective to achieve as high as possible order of convergence using a fixed number of function evaluations, which is closely connected to the optimal order of convergence in the sense of the Kung–Traub hypothesis. Kung and Traub [15] conjectured that multipoint methods without memory

* Corresponding author.
E-mail addresses: jrshira@yahoo.co.in (J.R. Sharma), iargyros@cameron.edu (I.K. Argyros), deepak.babbi@gmail.com (D. Kumar).

http://dx.doi.org/10.1016/j.jmaa.2016.12.051

based on $n+1$ function evaluations have the order of convergence at most 2^n. Multipoint methods with this property are called optimal methods.

The construction of multipoint iterative methods is mainly done by following two techniques, one is by using the weight functions and the second is by interpolation. The application of rational Hermite interpolation has been investigated by a number of authors including Ostrowski [19], Traub [29], Jarratt and Nudds [11] and Tornheim [28]. In particular, Ostrowski proposed a two-point method of optimal fourth order in which a rational function of order [1/1], i.e. a linear fraction

$$y(x) = \frac{(x-x_i)+a}{b(x-x_i)+c}, \tag{1}$$

is fitted at three points, two of which are coincident. Thus, a step in the iteration consists of matching f and y at two points x_i and w_i, where $w_i = x_i - f(x_i)/f'(x_i)$ is the Newton's point, and f' and y' at x_i only. The next approximation being given by zero of iteration function (1). In this way the following iterative method was obtained

$$\begin{cases} w_i = \ x_i - \frac{f(x_i)}{f'(x_i)}, \\ x_{i+1} = \ x_i - \frac{f(w_i)-f(x_i)}{f(w_i)f'(x_i)-f(x_i)f[x_i,w_i]} f(x_i), \quad i = 0,\, 1,\, 2, \dots \end{cases} \tag{2}$$

where x_0 is an initial approximation closer to a root (say, x^*) and $f[\cdot,\, \cdot]$ is first order divided difference. The error equation of this method is given as

$$e_{i+1} = A_2(A_2^2 - A_3)e_i^4 + O(e_i^5), \tag{3}$$

wherein $e_i = x_i - x^*$ and $A_k = (1/k!)f^{(k)}(x^*)/f'(x^*),\ k = 2, 3$.

In recent years, based on Ostrowski or Ostrowski-like optimal two-point fourth order methods many researchers have developed multipoint methods of optimal higher order of convergence using various techniques (see [3–7,9,10,13,14,16–18,20,23–27,30,31]). A more extensive list of references as well as a survey on progress made on the class of multipoint methods may be found in the recent book by Petković et al. [21].

Motivated by optimization considerations, here we derive a simple yet efficient class of n-point methods possessing optimal convergence order 2^n. The procedure is based on the simple application of rational approximants of different order at each step. Well-known classical Newton's and Ostrowski's methods are special cases of the class corresponding to $n = 1$ and $n = 2$. Eighth order and sixteenth order methods which correspond to $n = 3$ and $n = 4$ of the class are special cases of the eighth and sixteenth order methods proposed by Sharma et al. [25]. Analysis of the three-point eighth order and four-point sixteenth order methods finally pave the way for introducing the general n-point family. Numerical examples are considered to check the performance of new algorithms and to verify the theoretical results. Computational results including the elapsed CPU-time, confirm the efficient and robust character of the algorithms.

The rest of the paper is organized as follows. In Section 2, the three-point eighth order and four-point sixteenth order methods are presented and their convergence is discussed. The general n-point family is introduced in Section 3. Local convergence analysis of the general family is presented in Section 4. In Section 5, some numerical examples are considered to verify the theoretical results and to compare the performance of proposed schemes with some existing optimal methods. Concluding remarks are given in Section 6.

2. Optimal eighth and sixteenth order methods

In what follows, we will present the methods of optimal eighth and sixteenth order of convergence. The methodologies are based on Ostrowski's method (2) and further developed by using rational approximants of order $[1/(n-1)]$, $n = 3, 4$.

Please cite this article in press as: J.R. Sharma et al., On a general class of optimal order multipoint methods for solving nonlinear equations, J. Math. Anal. Appl. (2017), http://dx.doi.org/10.1016/j.jmaa.2016.12.051