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Multiple and nodal solutions for parametric Neumann problems with nonhomogeneous differential operator and critical growth ☆

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A B S T R A C T

We consider a parametric Neumann problem with nonhomogeneous differential operator and critical growth. Combining variational methods based on critical point theory, with suitable truncation techniques and flow invariance arguments, we show that for all large λ, the problem has at least three nontrivial smooth solutions, two of constant sign (one positive, the other negative) and the third nodal. We also study the asymptotic behavior of all solutions obtained when λ converges to infinity. The interesting point is that we do not impose any restrictions to the behavior of the nonlinear term f at infinity. Our work unifies and sharply improves several recent papers.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$, $1 < p < N$, and $p^\star = \frac{Np}{N-p}$. In this paper we study the following nonlinear parametric Neumann problem:

$$\begin{cases} -\mathrm{div} A(x, \nabla u) + \beta(x)\,|u|^{p-2}\,u = \lambda f(x,u) + g(x)\,|u|^{p^\star-2}\,u & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{P_λ}$$

where $A : \mathbb{R}^N \to \mathbb{R}^N$ is a continuous, strictly monotone map that satisfies certain regularity conditions. The precise hypotheses on the map are listed in the hypotheses H(a) below. These conditions incorporate in our framework many differential operators of interest such as the p-Laplacian. We stress that the differential

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operator need not be homogeneous and this is a source of difficulties, especially when we look for nodal (that is, sign changing) solutions. In problem (P_λ), $\lambda > 0$ is a parameter and in the reaction $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $g : \overline{\Omega} \to \mathbb{R}$ are assumed to be continuous functions with $g(x) \geq 0$ in $\overline{\Omega}$. Finally, we mention that the $\beta \in L^\infty(\Omega), \beta(x) \geq 0$ a.e. in $\Omega, \beta \neq 0$, and in the boundary condition n denotes the outward unit normal vector on $\partial\Omega$. Our aim is to prove a multiplicity theorem for problem (P_λ) providing sign information for all solutions provided $\lambda > 0$ is sufficiently large.

Equations driven by nonhomogeneous differential operators have been widely investigated in the subcritical case by variational methods under Dirichlet [1,17,31–33], Neumann [2,13,16,19,23,25,29], or Robin [28] boundary condition. We mention that the works [2,13,17,19,23,29,32,33] produce nodal solutions. On the other hand, it is generally hard to handle nonlinear nonhomogeneous equations without the subcritical growth condition, and thus, the results in the direction are very rare (see [18,26,30]). In [18,30], the right hand side nonlinearity is assumed to be odd near zero, and the authors produced a whole sequence of distinct nodal solutions. Based on variational methods combining invariant sets of descending flow, Motreanu and Tanaka [26] obtained the existence of a positive solution, a negative solution and a sign-changing solution for equation (P_λ) with $\beta = g = 0$ if $\lambda > 0$ is sufficiently large. They assumed that problem (P_λ) admits an ordered pair of super and lower solution. In the present paper we prove a similar three-solutions-theorem for problem (P_λ) providing sign information for all solutions obtained. Moreover, we obtain the asymptotic behavior of the three solutions when λ converges to infinity. The interesting feature of our work here, is that in problem (P_λ) the nonlinearity f satisfies a superlinear growth condition just in a neighborhood of zero. By using variational methods together with suitable truncation techniques and flow invariance arguments, we are able to avoid restrictions on the behavior of the nonlinearity f at infinity. Then we can handle nonlinearities $f(x, u)$ containing terms like $|u|^{r-2} u$ and $|u|^{r-2} ue^u$ with $p < r < \infty$.

Throughout this paper, we assume that the map A and the function f satisfy the following hypotheses H(a) and H(f), respectively:

H(a). $A(x, y) = h(x, |y|)y$, where $h(x, t) > 0$ for all $(x, t) \in \overline{\Omega} \times (0, +\infty)$, and

(i) $A \in C^{0,\epsilon}_{\text{loc}}(\overline{\Omega}\times\mathbb{R}^N, \mathbb{R}^N) \cap C^1(\overline{\Omega} \times \mathbb{R}^N \setminus \{0\}, \mathbb{R}^N)$ with some $0 < \epsilon \leq 1$;
(ii) there exist constants $C_1 > 0$ and $1 < p < +\infty$ such that

$$\left|\nabla_y A(x, y)\right| \leq C_1 |y|^{p-2} \quad \text{for every } x \in \overline{\Omega}, \text{ and } y \in \mathbb{R}^N \setminus \{0\};$$

(iii) there exists $C_0 > 0$ such that

$$(\nabla_y A(x, y)\xi, \xi)_{\mathbb{R}^N} \geq C_0 |y|^{p-2} |\xi|^2 \quad \text{for every } x \in \overline{\Omega},\ y \in \mathbb{R}^N \setminus \{0\} \text{ and } \xi \in \mathbb{R}^N;$$

(iv) for all $(x, y) \in \overline{\Omega} \times \mathbb{R}^N$, we have

$$pG(x, y) \geq (A(x, y), y)_{\mathbb{R}^N},$$

where $G(x, y)$ is the primitive of $A(x, y)$, i.e., $\nabla_y G(x, y) = A(x, y)$ for every $x \in \overline{\Omega}, y \in \mathbb{R}^N$, and $G(x, 0) = 0$.

In the above hypotheses by $|\cdot|$ we denote the Euclidean norm in $\mathbb{R}^N$. And the notation $\nabla_y A$ means the differential of the mapping $A(x, y)$ with respect to the variable $y \in \mathbb{R}^N$. Similar conditions are used widely in the literature (see, e.g., [1,9,24,25,31]).

H(f). $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous function with primitive $F(x, t) = \int_0^t f(x, s)\mathrm{d}s$ satisfying $f(x, 0) = 0$ for a.a. $x \in \Omega$ and