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Variational inequalities of Navier–Stokes type with time dependent constraints

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ABSTRACT

We consider a class of parabolic variational inequalities with time dependent obstacle of the form $|\boldsymbol{u}(x,t)| \leq p(x,t)$, where $\boldsymbol{u}$ is the velocity field of a fluid governed by the Navier–Stokes variational inequality. The obstacle function $p = p(x,t)$, imposed on $\boldsymbol{u}$, consists of three parts, which are respectively: the degenerate part $p(x,t) = 0$, the finitely positive part $0 < p(x,t) < \infty$ and the singular part $p(x,t) = \infty$. In this paper, we shall propose a sequence of approximate obstacle problems with everywhere finitely positive obstacles, and prove an existence result for the original problem by discussing convergence of the approximate problems. The crucial step is to handle the nonlinear convection term. In this paper we propose a new approach to it.

1. Introduction

In real problems, we find many dynamical processes which occur in fluids or in consequence of a fluid flow. Their mathematical models include then a hydrodynamic equation, typically Stokes or Navier–Stokes, coupled with some other evolution systems, such as heat convection [14,16], phase transitions [1] or biofilm growth [11,23]. These couplings may have the form of transport or advection, but they may also mean some evolution of the domain in which the flow takes place. To give just one example of phenomenon of importance to medicine and in which both types of couplings appear at the same time scale, let us have a look on the mentioned biomass growth. In a fluid transporting some living organisms and some appropriate nutrient, some of these organisms can stick to the boundary of the fluid flow's domain (e.g. blood vessels walls) and then aggregate, in which way they gradually restrict the domain available for the flow, forming a geometrical obstacle to it.

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Mathematical analysis of models for such systems seems not easy, since the theory on partial differential systems coupled with equations, or variational inequalities, of the Navier–Stokes type has not been completely established.

In this paper, we address the problem of a Navier–Stokes flow constrained by some evolving in time obstacle. We model the obstacle as a non-negative function p, depending on the space and time variable, which is a bound imposed *a priori* on the velocity of the flow. The Navier–Stokes equation becomes then naturally a variational inequality. We allow the constraint to disappear ($p = \infty$, free flow), to be a total obstacle ($p = 0$, no flow) or only partial ($0 < p < \infty$). We assume that p is continuous. Our main result is Theorem 1.1 below, stating existence and some regularity of solution to this problem.

This kind of parabolic obstacle problem would be useful for mathematical modelling of various nonlinear problems in hydrodynamic fluids, see e.g. [2,9–11,18,19,21]. As far as variational inequalities of Navier–Stokes type are concerned, see e.g. [3–6,24,25] for a constant in time constraint, and [13], where the constraint can be time and space dependent. However, even this last case did not allow the "free flow" and "no-flow" regions, i.e. the obstacle function p had to be finite and bounded from below by a positive constant — a serious limitation of the model that we overcome in the present work. It is clear that especially allowing the "total obstacle" case, i.e. having regions where $p = 0$, is essential from the point of view of modelling; it is also the main challenge for the mathematical analysis that we are presenting.

For basic studies on Navier–Stokes equations and phase transitions, we refer to [27] and [8], respectively. Our formulation of the Navier–Stokes inequality arising from the obstacle, that we state in Definition 1.1 below, is analogous to these appearing in [3–6,24,25]. For its analysis, we will use the theory of subdifferentials contained in [7,17,22,28]. This will be exposed in Section 2.

Let us set the basic functional framework and explicit the assumptions so as to formulate the main result. Let Ω be a bounded domain in $\mathbf{R}^3$ with smooth boundary $\Gamma := \partial\Omega$, $Q := \Omega \times (0, T)$, $0 < T < \infty$ and $\Sigma := \Gamma \times (0, T)$, and denote by $|\cdot|_X$ the norm in various function spaces X built on Ω as well as by $\|\cdot\|_Y$ for function spaces Y on $\Omega \times (0, T)$. Also, consider the usual solenoidal function spaces:

$$\begin{aligned}
&\boldsymbol{\mathcal{D}}_\sigma(\Omega) := \{\boldsymbol{v} = (v^{(1)}, v^{(2)}, v^{(3)}) \in \mathcal{D}(\Omega)^3 \mid \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega\},\\
&\boldsymbol{H}_\sigma(\Omega) := \text{the closure of } \boldsymbol{\mathcal{D}}_\sigma(\Omega) \text{ in } L^2(\Omega)^3, \text{ with norm } |\cdot|_{0,2},\\
&\boldsymbol{V}_\sigma(\Omega) := \text{the closure of } \boldsymbol{\mathcal{D}}_\sigma(\Omega) \text{ in } H^1_0(\Omega)^3, \text{ with norm } |\cdot|_{1,2},\\
&\boldsymbol{W}_\sigma(\Omega) := \text{the closure of } \boldsymbol{\mathcal{D}}_\sigma(\Omega) \text{ in } W^{1,4}_0(\Omega)^3, \text{ with norm } |\cdot|_{1,4};
\end{aligned}$$

in these spaces the norms are given as usual by

$$|\boldsymbol{v}|_{0,2} := \left\{ \sum_{k=1}^{3} \int_\Omega |v^{(k)}|^2 dx \right\}^{\frac{1}{2}}, \quad |\boldsymbol{v}|_{1,2} := \left\{ \sum_{k=1}^{3} \int_\Omega |\nabla v^{(k)}|^2 dx \right\}^{\frac{1}{2}}$$

and

$$|\boldsymbol{v}|_{1,4} := \left\{ \sum_{k=1}^{3} \int_\Omega |\nabla v^{(k)}|^4 dx \right\}^{\frac{1}{4}}.$$

For simplicity we denote the dual spaces of $\boldsymbol{V}_\sigma(\Omega)$ and $\boldsymbol{W}_\sigma(\Omega)$ by $\boldsymbol{V}^*_\sigma(\Omega)$ and $\boldsymbol{W}^*_\sigma(\Omega)$, respectively, which are equipped with their dual norms $|\cdot|_{-1,2}$ and $|\cdot|_{-1,\frac{4}{3}}$. Also, we denote the inner product in $\boldsymbol{H}_\sigma(\Omega)$ by

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