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Simultaneous zero inclusion property for spatial numerical ranges

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A R T I C L E I N F O

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For a finite dimensional complex normed space $\mathscr{X}$, we say that it has the simultaneous zero inclusion property if an invertible linear operator S on $\mathscr{X}$ has zero in its spatial numerical range if and only if zero is in the spatial numerical range of the inverse S^{-1}, as well. We show that beside Hilbert spaces there are some other normed spaces with this property. On the other hand, space $\ell_1(n)$ does not have this property. Since not every normed space has the simultaneous zero inclusion property, we explore the class of invertible operators at which this property holds. In the end, we consider a property which is stronger than the simultaneous zero inclusion property and is related to the question when it is possible, for every invertible operator S, to control the distance of 0 to the spatial numerical range of S^{-1} by the distance of 0 to the spatial numerical range of S.

1. Introduction

Let $\mathscr{X}$ be a finite dimensional complex normed space. In this paper, we identify $\mathscr{X}$ with $(\mathbb{C}^n, \|\cdot\|)$, where n is the dimension of $\mathscr{X}$. By $\mathscr{S}_{\mathscr{X}}$, respectively $\mathscr{B}_{\mathscr{X}}$, we denote the unit sphere, respectively the closed unit ball, in $\mathscr{X}$. The algebra $\mathcal{L}(\mathscr{X})$ of all linear operators on $\mathscr{X}$ is identified with $(\mathbb{M}_n, |\!|\!| \cdot |\!|\!|)$, the algebra of all $n \times n$ complex matrices endowed with the associated operator norm $|\!|\!| S |\!|\!| = \sup\{\|Sx\|;\ x \in \mathscr{S}_{\mathscr{X}}\}$. The dual space of $\mathscr{X}$ is $\mathscr{X}' = (\mathbb{C}^n, \|\cdot\|^d)$; the pairing between $\mathscr{X}$ and $\mathscr{X}'$ is standard: if $x = (x_1, \ldots, x_n)^\mathsf{T} \in \mathscr{X}$ and $\xi = (\xi_1, \ldots, \xi_n)^\mathsf{T} \in \mathscr{X}'$, then $\langle x, \xi \rangle = x_1\xi_1 + \cdots + x_n\xi_n$, and the dual norm $\|\cdot\|^d$ on $\mathscr{X}'$ is given by $\|\xi\|^d = \sup\{|\langle x, \xi\rangle|;\ x \in \mathscr{S}_{\mathscr{X}}\}$.

For every $x \in \mathscr{S}_{\mathscr{X}}$, let $\mathscr{D}(x) = \{\xi \in \mathscr{S}_{\mathscr{X}'};\ \langle x, \xi \rangle = 1\}$. By the Hahn–Banach theorem, $\mathscr{D}(x)$ is a non-empty subset of $\mathscr{S}_{\mathscr{X}'}$. Let $\Pi(\mathscr{X}) = \{(x, \xi) \in \mathscr{S}_{\mathscr{X}} \times \mathscr{S}_{\mathscr{X}'};\ \xi \in \mathscr{D}(x)\}$. The spatial numerical range of $S \in \mathcal{L}(\mathscr{X})$ is

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$$W(S) = \{\langle Sx, \xi\rangle; \quad (x,\xi) \in \Pi(\mathscr{X})\} \tag{1.1}$$

(see [1–3]). If we want to stress the underlying normed space, then we write $W_{\mathscr{X}}(S)$ instead of $W(S)$. Since $\mathscr{X}$ is finite dimensional, $\Pi(\mathscr{X})$ is a compact subset of $\mathscr{S}_{\mathscr{X}} \times \mathscr{S}_{\mathscr{X}'}$ and the spatial numerical range is its image under a continuous bilinear map. Hence $W(S)$ is a non-empty compact subset of $\mathbb{C}$.

In the case when the normed space is a (finite dimensional) Hilbert space $\mathscr{H}$ with inner product $(\cdot \mid \cdot)$, then $\Pi(\mathscr{H}) = \{(x,x);\ x \in \mathscr{S}_{\mathscr{H}}\}$ and the spatial numerical range of S is equal to the classical numerical range $W_{\mathscr{H}}(S) = \{(Sx \mid x); \quad x \in \mathscr{S}_{\mathscr{H}}\}$, known in the literature also as the field of values of S. Note that in this case the dual of $\mathscr{H}$ is identified with $\mathscr{H}$ through the inner product which is sesquilinear; a vector $y \in \mathscr{H}$ is identified with the functional $x \mapsto (x \mid y)$ $(x \in \mathscr{H})$. We refer the reader to [5,8] for more information about numerical ranges of operators on Hilbert spaces. Here we mention Toeplitz–Hausdorff Theorem which says that $W_{\mathscr{H}}(S)$ is a convex subset of $\mathbb{C}$. However the spatial numerical range of an operator on a general complex normed spaces is not necessary convex; see [2, Example 6 in §21] and also Example 3.4 below.

The starting point of this paper is the following simple observation about numerical ranges of operators on a complex Hilbert space (finite or infinite dimensional). If S is an invertible bounded linear operator on $\mathscr{H}$ such that $0 \in W_{\mathscr{H}}(S)$, then $0 \in W_{\mathscr{H}}(S^{-1})$. Indeed, let $x \in \mathscr{S}_{\mathscr{H}}$ be such that $(Sx \mid x) = 0$. Then $y = \frac{1}{\|Sx\|}Sx$ is a vector in $\mathscr{S}_{\mathscr{H}}$ and $(S^{-1}y \mid y) = \frac{1}{\|Sx\|^2}(x \mid Sx) = 0$, that is, $0 \in W_{\mathscr{H}}(S^{-1})$. As we shall see in Section 3, Example 3.7, not every normed space has this property.

Definition 1.1. A normed space $\mathscr{X} = (\mathbb{C}^n, \|\cdot\|)$ has the *simultaneous zero inclusion property*, briefly S0I property, if the equivalence $0 \in W(S) \iff 0 \in W(S^{-1})$ holds for every invertible $S \in \mathcal{L}(\mathscr{X})$.

The aim of this paper is to explore the S0I property and related properties of norms on $\mathbb{C}^n$. In Section 2 we show that beside Hilbert spaces there are some other normed spaces with the S0I property. However, all examples which we know are derived from Hilbert spaces. Not every normed space has the S0I property. In order to present an explicit example of a normed space without the S0I property, we explore in Section 3 spatial numerical ranges of operators on $\ell_1(n)$; in the case $n = 2$ we are able to give a complete description. This allows us to show that there exist invertible operators S on $\ell_1(n)$ such that $0 \in W_{\ell_1(n)}(S)$ and $0 \notin W_{\ell_1(n)}(S^{-1})$. The same holds for the space $\ell_\infty(n)$. We believe that for any $1 < p \neq 2 < \infty$ the space $\ell_p(n)$ does not have the S0I property. Since not every normed space has the S0I property we are interested in those invertible operators S at which the S0I property holds locally, i.e., $0 \in W(S)$ if and only if $0 \in W(S^{-1})$. The related results are collected in Section 4. In the last section we consider a property which is stronger than the S0I property. Namely, we are concerned with the question when it is possible, for every invertible operator S, to control the distance of 0 to the spatial numerical range of S^{-1} by the distance of 0 to the spatial numerical range of S.

2. Spaces with the S0I property

Let $\mathscr{X} = (\mathbb{C}^n, \|\cdot\|)$ be a normed space and let $T \in \mathbb{M}_n$ be invertible. Then the mapping $\|\cdot\|_T : x \mapsto \|Tx\|$ $(x \in \mathbb{C}^n)$ defines a norm on $\mathbb{C}^n$, see [7, Theorem 5.3.2]. We denote by $\mathscr{X}_T$ the normed space $(\mathbb{C}^n, \|\cdot\|_T)$.

Lemma 2.1. *Let $\mathscr{X} = (\mathbb{C}^n, \|\cdot\|)$ be a normed space, $T \in \mathbb{M}_n$ be invertible, and $\xi \in \mathscr{X}'$, $S \in \mathbb{M}_n$ be arbitrary. Then*

(i) $\mathscr{S}_{\mathscr{X}_T} = T^{-1}\big(\mathscr{S}_{\mathscr{X}}\big)$ *and* $\mathscr{B}_{\mathscr{X}_T} = T^{-1}\big(\mathscr{B}_{\mathscr{X}}\big)$;
(ii) $\|\xi\|_T^d = \|(T^{-1})^{\mathsf{T}}\xi\|^d$;
(iii) $\| S \|_T = \| TST^{-1} \|$;
(iv) $W_{\mathscr{X}_T}(S) = W_{\mathscr{X}}(TST^{-1})$.