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Global behaviour of the period function of the sum of two quasi-homogeneous vector fields

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ABSTRACT

We study the global behaviour of the period function on the period annulus of degenerate centres for two families of planar polynomial vector fields. These families are the quasi-homogeneous vector fields and the vector fields given by the sum of two quasi-homogeneous Hamiltonian ones. In the first case we prove that the period function is globally decreasing, extending previous results that deal either with the Hamiltonian quasi-homogeneous case or with the general homogeneous situation. In the second family, and after adding some more additional hypotheses, we show that the period function of the origin is either decreasing or has at most one critical period and that both possibilities may happen. This result also extends some previous results that deal with the situation where both vector fields are homogeneous and the origin is a non-degenerate centre.

1. Introduction and main results

A planar polynomial vector field $X(x,y) = (P(x,y), Q(x,y))$ is called (p,q) quasi-homogeneous of quasi-degree n if there exist $p, q, n \in \mathbb{N}$ such that

$$P(\lambda^p x, \lambda^q y) = \lambda^{n+p-1} P(x,y), \quad Q(\lambda^p x, \lambda^q y) = \lambda^{n+q-1} Q(x,y),$$

for all $\lambda \in \mathbb{R}$. It is not restrictive to take p and q coprime. The numbers p and q are usually called weights. It is well known that its associated differential equation

$$\begin{cases} \dot{x} = P(x,y), \\ \dot{y} = Q(x,y), \end{cases}$$

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can be integrated by writing it in the so called generalized polar coordinates, see for instance [21] or Section 2. Notice that homogeneous vector fields of degree n are quasi-homogeneous of quasi-degree n and weights $(1, 1)$. Moreover, in this case the generalized polar coordinates are the usual polar ones.

In this paper we are concerned with vector fields having a degenerate critical point at the origin of centre type, and being either quasi-homogeneous or given by the sum of two quasi-homogeneous ones sharing the same weights (p, q). In the latter case, additionally we will assume that the vector field is Hamiltonian. We write the vector field in both situations as $X(x, y) = X_n(x, y) + X_m(x, y)$, with associated differential equation

$$\begin{cases} \dot{x} = P_n(x,y) + P_m(x,y), \\ \dot{y} = Q_n(x,y) + Q_m(x,y), \qquad 1 < n < m, \end{cases} \tag{1.1}$$

where each $X_j = (P_j, Q_j)$, $j \in \{n, m\}$, is (p, q) quasi-homogeneous of quasi-degree j. We assume that $X_n(x, y) \not\equiv 0$ but we admit that $X_m(x, y) \equiv 0$.

We want to know the global behaviour of the period function on the period annulus of the origin when we assume that the differential equation associated to X has a degenerate centre at this point. Recall that a *centre* is a critical point that has a punctured neighbourhood full of periodic orbits. The largest of such neighbourhoods is called the *period annulus* of the centre. When the eigenvalues of DX at the centre are not purely imaginary, then the centre is called *degenerate*. This is our situation because $n > 1$. The function that associates to any closed curved of the period annulus its period is called the *period function* of the centre. It is well known that the period function tends to infinity when the orbits in a period annulus approach either to a degenerate centre or to a polycycle with some finite critical point, see for instance [9].

In general, given a system with a centre, we will write $T(x, y)$ to denote the period of the orbit passing through the point (x, y). When the system is Hamiltonian, it is sometimes more convenient to parameterize the periodic orbits by their energy h and write $T(h)$ to denote their corresponding periods. The *critical periods* are the zeroes of the derivative of the period function once the continuum of periodic orbits is parameterized by a smooth one-parameter function. This parameter can be the energy in the Hamiltonian situation, or anyone describing a transversal section to the orbits. It is not difficult to prove that the number of critical periods does not depend neither on the transversal section, nor on its parametrization. When a zero of the derivative of the period function is simple we will say that the system has a *simple* critical period.

Some motivations to know properties of the period function come from its role in the study of several differential equations. For instance, it appears in mathematical models in physics or ecology, see [15,17,23, 24] and the references therein. From a more mathematical point of view, it is important in the study of the bifurcations from a continuum of periodic orbits giving rise to isolated ones, see [8, pp. 369–370], in the description of the dynamics of some discrete dynamical systems, see [4,11,12] or for counting the solutions of some boundary value problems, see [6,7].

The period function for homogeneous vector fields (both Hamiltonian and non-Hamiltonian) was characterized in [14], while the quasi-homogeneous Hamiltonian were studied in [25]. Our main result for the quasi-homogeneous case, i.e. $X_m(x, y) \equiv 0$, completely characterizes the period function in the general case, extending their results.

Theorem A. *Consider a (p, q) quasi-homogeneous vector field of quasi-degree n, that is (1.1) with $X_m = 0$, with a degenerate centre at the origin. Then its associated period function is monotonic decreasing. Moreover it can be written as*

$$T(\xi, 0) = T_1\,\xi^{\frac{1-n}{p}}, \qquad or \qquad T(0, \eta) = T_2\,\eta^{\frac{1-n}{q}},$$

for $\xi, \eta \in \mathbb{R}^+$, and some non-zero constants T_1 and T_2.