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Stability by linear approximation for time scale dynamical systems

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A B S T R A C T

We study systems on time scales that are generalizations of classical differential or difference equations and appear in numerical methods. In this paper we consider linear systems and their small nonlinear perturbations. In terms of time scales and of eigenvalues of matrices we formulate conditions, sufficient for stability by linear approximation. For non-periodic time scales we use techniques of central upper Lyapunov exponents (a common tool of the theory of linear ODEs) to study stability of solutions. Also, time scale versions of the famous Chetaev's theorem on conditional instability are proved. In a nutshell, we have developed a completely new technique in order to demonstrate that methods of non-autonomous linear ODE theory may work for time-scale dynamics.

1. Introduction

We study dynamic equations on time scales i.e. on unbounded closed subsets of $\mathbb{R}$. The time scale approach first introduced by S. Hilger and his collaborators (see [1] and references therein) was intensively developing during last decades. The first advantage of such approach is the common language that fits both for flows and diffeomorphisms. On the other hand, there are many numerical methods that correspond to non-uniform steps. Especially, this is applicable for modeling non-smooth or strongly non-linear dynamical systems.

Consider a motion of a particle in two distinct media, e.g. water and air. Evidently, to model such system, it is not effective to use equidistant nodes. It is better to take more of them inside time periods, corresponding

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to motions in water. This is a natural way to obtain a non-trivial time scale in a real life problem (see [29] and references therein). Another application of time scale analysis may be found for systems with delay [5].

In this sense it seems to be useful to generalize some results on stability theory well-known for ODEs for time scale case. Mainly, we consider a general linear system (autonomous or non-autonomous) and its small non-linear perturbation. For the continuous dynamics, there exists a well-developed theory on stability by first approximation. For autonomous case there are classical stability criteria related to eigenvalues of a matrix of coefficients for linear approximation (call it $\mathcal{A}$).

For non-autonomous systems or for cases when eigenvalues do not give information on stability of the perturbed system, there might be two approaches. The first one is based on the theory of Lyapunov functions (see [21] and references therein for review on the time scale version of this method). The second one involves integral inequalities, particularly the Grönwall–Bellman inequality (see [4,6]). For ordinary differential equations there is a very powerful tool that allows to find stability of solutions via the so-called central upper Lyapunov exponents [10]. In this paper we combine all referred methods in order to study time scale dynamics.

Exponential stability for solutions of time-varying dynamic equations on a time scale have been investigated by many authors. We mention recent papers by Bohner and Martynyuk [7] (this article is also a good introduction to the theory of time scale systems), Du and Tien [15], Hoffacker and Tisdell [18] and Martynyuk [24]. We also refer to papers [16,19,20,23,26] where related problems are studied and new approaches have been introduced.

A "multidimensional" analog of time scales called discrete differential geometry is also studied, see [3] and references therein. In such problems, time scales may appear, for instance, as discretizations of geodesic flows.

However, the following problems were open by now.

1. For constant matrices $\mathcal{A}$, are there any stability criteria for perturbed time-scale systems?
2. Is there any analog of Chetaev's theorem on instability by the first approximation for time scale systems?
3. Are there any sufficient conditions on stability by the first approximation, close to necessary ones?

One of principal difficulties in the theory of time scale systems is that, generally speaking, in "autonomous" case (i.e. when the right hand side of the system does not depend on t) the system does not define a flow (a shift of a solution is not necessarily a solution, group property may be violated etc). Also, one must carefully check basic properties like smoothness of solutions that can be violated even for systems with smooth right hand sides.

In our paper we give positive answers to all mentioned questions. The main idea of our paper is very simple: methods of classical theory of linear non-autonomous differential equations are applicable for time scale systems. Here we notice that in time scale analysis there are two types of derivatives: the so-called Δ- and ∇-derivatives (see [7] for details). In this paper we study Δ-derivatives only. However, it seems that the main ideas of our work can be easily transferred to equations with ∇-derivatives.

We have two principal objectives.

First, we provide sufficient conditions on stability by the first approximation. We demonstrate that the obtained conditions are close to necessary ones. In our proofs, we use the techniques of central upper Lyapunov exponents. This approach seems to be novel for time scale analysis. We can use many related tools such as Millionschikov's rotations to obtain instability.

Secondly, we prove an analog of Chetaev's theorem on instability by first approximation. Specifics of time scales demands a novel, non-classical approach to proof since, generally speaking, we cannot use tools of the theory of autonomous systems, anymore.

The paper is organized as follows. In Section 2 we give a brief introduction to time scale analysis. In Section 3 we give a review of existing results on stability of time scale equations. In these two sections