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# Bessel orbits of normal operators

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Given a bounded normal operator $A$ in a Hilbert space and a fixed vector $x$, we elaborate on the problem of finding necessary and sufficient conditions under which $(A^k x)_{k\in\mathbb{N}}$ constitutes a Bessel sequence. We provide a characterization in terms of the measure $\|E(\cdot)x\|^2$, where $E$ is the spectral measure of the operator $A$. In the separately treated special cases where $A$ is unitary or selfadjoint we obtain more explicit characterizations. Finally, we apply our results to a sequence $(A^k x)_{k\in\mathbb{N}}$, where $A$ arises from the heat equation. The problem is motivated by and related to the new field of Dynamical Sampling which was recently initiated by Aldroubi et al. in [3].

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## 1. Introduction

Many signals in nature obey a differential equation of the type $\frac{\partial f}{\partial t} = Tf$ with a linear operator $T$. Dynamical Sampling incorporates this knowledge aiming to recover $f$ from spatial subsamples at several times. A prominent example is that of sensors in a forest measuring the temperature in order to prevent or detect forest fire. Usually, many sensors are needed to retrieve an accurate temperature distribution at a specific point of time. Making use of the heat equation as the time evolution law in the background, the hope is to install less sensors but to measure sequentially at different times, which is – of course – economically more efficient.

Let us briefly motivate the general Dynamical Sampling setting. Assume that $T$ is a bounded operator in a Hilbert space $\mathcal{H}$ of functions. If we put $u(t) := f(t,\cdot)$, the differential equation reduces to $\dot{u}(t) = Tu(t)$ and has the solutions $u(t) = e^{tT}u_0$. Hence, sampling the functions $f(t,\cdot)$ at times $t = 0, 1, 2, \dots$ is the same as sampling $u_0$, $Bu_0$, $B^2u_0$, etc., where $B = e^T$. Here, we assume that "sampling" $g \in \mathcal{H}$ means to compute scalar products between $g$ and functions $x_i$, $i \in I$, from a fixed system.[1] Hence, the measurements have the

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[1] If $\mathcal{H}$ is a reproducing kernel Hilbert space with reproducing kernel $x_t$, then this type of sampling obviously coincides with function evaluation.

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form $\langle u_0, A^k x_i\rangle$, where $A = B^*$. Therefore, recovering $u_0$ from these measurements in a stable way requires the system $(A^k x_i)_{i\in I,\, k=0,\dots,K_i}$ to be a frame for some numbers $K_i \in \mathbb{N}\cup\{\infty\}$, $i \in I$. Dynamical Sampling is about finding necessary and sufficient conditions on the operator $A$, the vectors $x_i$, and the numbers $K_i$ under which $(A^k x_i)_{i\in I,\, k=0,\dots,K_i}$ is a frame for $\mathcal{H}$.

Motivated by works of Vetterli et al. (see [13,15,18]), Aldroubi, Davis, and Krishtal first considered the case where $B$ is a convolution operator [4] (see also [1]). The general Dynamical Sampling problem was tackled in the paper [3], recently followed by the successor [2] (see also [5]). In [3] the authors completely describe the finite-dimensional situation and also characterize the frame property of $(A^k x)_{k\in\mathbb{N}}$ in the special case where $A$ is a selfadjoint operator with eigenvectors forming an orthonormal basis of $\mathcal{H}$. The paper [2] deals with normal operators $A$ and provides several necessary and sufficient conditions under which $(A^k x_i)_{i\in I,\, k=0,\dots,K_i}$ is minimal or complete. The conditions are formulated in terms of a certain decomposition of the normal operator $A$, taking into account the spectral multiplicity of $A$.

The present paper is motivated by the following simple observation: in order that $(A^k x_i)_{i\in I,\, k=0,\dots,K_i}$ be a frame for $\mathcal{H}$ it is necessary that each of the systems $(A^k x_i)_{k=0,\dots,K_i}$, $i \in I$, is a Bessel sequence. As this is only interesting for those $i \in I$ with $K_i = \infty$, we consider systems of the form $(A^k x)_{k\in\mathbb{N}}$ and ask for which normal operators $A$ and which vectors $x$ the corresponding system is a Bessel sequence. It quickly turns out that the answer to this question solely depends on the properties of the measure $\mu_x := \|E(\cdot)x\|^2$, where $E$ is the spectral measure of the normal operator $A$: it is shown in Lemma 4.1 that $(A^k x)_{k\in\mathbb{N}}$ is a Bessel sequence if and only if the sequence $(z^k)_{k\in\mathbb{N}}$ of monomials is a Bessel sequence in $L^2(\mu_x)$.

In the case where $A$ is unitary we find a comparatively simple and explicit characterization (see Theorem 3.2): The system $(A^k x)_{k\in\mathbb{N}}$ is a Bessel sequence if and only if the measure $\mu_x$ is Lipschitz continuous (see Definition 3.1) with respect to the arc length measure on the unit circle $\mathbb{T}$. For selfadjoint operators $A$ we provide two characterizations in Theorem 3.5 one of which is even more simple. It states that $(A^k x)_{k\in\mathbb{N}}$ is a Bessel sequence if and only if $\langle A^k x, x\rangle = O(k^{-1})$ as $k \to \infty$. In the proof we make use of results from the theory of Hankel matrices. In the case of general normal operators $A$ it is necessary for $(A^k x)_{k\in\mathbb{N}}$ to be a Bessel sequence that the support of the measure $\mu_x$ lies in the closed unit disc $\overline{\mathbb{D}}$ (see Lemma 2.2). The Bessel sequence property of $(A^k x)_{k\in\mathbb{N}}$ is now highly dependent on the behavior of the measure $\mu_x$ close to the unit circle line. In fact, we prove in Theorem 4.5 that $(A^k x)_{k\in\mathbb{N}}$ is a Bessel sequence if and only if the support of $\mu_x$ lies in $\overline{\mathbb{D}}$, $\mu_x|\mathbb{T}$ is Lipschitz continuous with respect to the arc length measure and $\mu_x|\mathbb{D}$ is a Carleson measure. This theorem also provides a characterization in terms of resolvent growth. In the end of the paper we return to the Dynamical Sampling problem in connection with the diffusion operator in the heat equation and prove the negative result that in this case the resulting sequence is not a Bessel sequence.

The paper is organized as follows. In Section 2 we fix our notation and recall some notions that are used in the paper. In Section 3 we present characterizations for $(A^k x)_{k\in\mathbb{N}}$ to be a Bessel sequence in the case of unitary and selfadjoint operators $A$. After that we characterize the Bessel sequence property of $(A^k x)_{k\in\mathbb{N}}$ for general normal operators $A$ in Section 4 and provide some sufficient conditions in Section 5. In Section 6 we show that Dynamical Sampling in a certain framework connected to the heat equation is not possible when the function is sampled at an infinite amount of times. The last section, Section 7, is an appendix containing auxiliary results from measure theory and operator theory which are used in Sections 3–6.

## 2. Notation, preliminaries, and setting

By $\mathbb{N}$ we denote the natural numbers *including zero*, whereas $\mathbb{N}^* := \mathbb{N}\setminus\{0\}$. The Borel $\sigma$-algebra on $\mathbb{C}$ is denoted by $\mathfrak{B}$. If $\Delta_0 \in \mathfrak{B}$ we set $\mathfrak{B}(\Delta_0) := \{\Delta \in \mathfrak{B} : \Delta \subset \Delta_0\}$. By $\mathbb{T}$ we denote the unit circle and by $\mathbb{D}$ the open unit disc in $\mathbb{C}$. Recall that the scalar product on $L^2(\mathbb{T})$ is given by $\langle f, g\rangle = \int_{\mathbb{T}} f(z)\overline{g(z)}\,d|z|$, where the arc length measure arc $:= |\cdot|$ is such that $\int_{\mathbb{T}} d|z| = 1$. By $\mathrm{Leb}_n$ we denote the Lebesgue measure on $\mathbb{R}^n$. For $z \in \mathbb{C}$ and $k \in \mathbb{Z}$ we set

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