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Sharp Hardy inequalities on the solid torus

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A R T I C L E I N F O

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A B S T R A C T

In this paper, we establish the classical Hardy inequality in the solid torus and some variants of it. The general idea is to use the fact that Sobolev embeddings can be improved in the presence of symmetries. In all cases, using techniques that exploit the symmetry presented by the solid torus, we calculate the displayed best constants and we prove that they are the same as the standard Hardy best constants which appear in convex domains although the solid torus is not convex.

1. Introduction and a short overview

The classical Hardy inequality was established by Hardy in 1920's and in the continuous form it informs us that:

If $1 < p < \infty$ and f is a non-negative p-integrable function on $(0, \infty)$, then f is integrable over the interval $(0, x)$ for each positive x and

$$\int_0^\infty \left(\frac{1}{x} \int_0^x f(t)\, dt \right)^p dx \leqslant \left(\frac{p}{p-1} \right)^p \int_0^\infty f^p(x)\, dx. \tag{1}$$

The constant $\left(\frac{p}{p-1}\right)^p$ in (1) is sharp, i.e. it cannot be replaced by a smaller number so that (1) remains true for all relevant functions, respectively, and equality holds only if $f = 0$.

The inequality (1) was established by Hardy in [21] and was first highlighted in the famous book [23] of Hardy, Littlewood, and Polya or in the original article of Hardy [22], who also showed that the constant is not attained, i.e. the variational problem has no minimizer. As known, inequality (1) is the standard form of the large family of the Hardy and Hardy-type inequalities which constitute an essential tool in Analysis,

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in the study of PDE's, and in the Calculus of variations. In addition, we can find various applications in Geometry, in Mathematical Physics and in Astrophysics.

A proof of the above inequality was given by Landau, in a letter to Hardy, which officially was published in [29]. For a short but very informative presentation of prehistory of Hardy's inequality, see in [28].

Coming back to the inequality (1), if we set $u(x) = \int_0^x f(t)\, dt$, we obtain the inequality

$$\int_0^\infty \left(\frac{u(x)}{x}\right)^p dx \leqslant \left(\frac{p}{p-1}\right)^p \int_0^\infty \left(u'(x)\right)^p dx, \tag{2}$$

which is the most popular form of the classical Hardy inequality in the contemporary literature.

The following Hardy inequality is the classical generalization of Hardy inequality (1) to higher dimensions and according to which for $n > 1$, $1 \le p < \infty$ with $p \neq n$ and for all $u \in C_0^\infty(\mathbb{R}^n \backslash \{0\}$, it holds

$$\int_{\mathbb{R}^n} \frac{|u(\mathbf{x})|^p}{|\mathbf{x}|^p} d\mathbf{x} \leqslant \left|\frac{p}{n-p}\right|^p \int_{\mathbb{R}^n} |\nabla u(\mathbf{x})|^p\, d\mathbf{x}, \tag{3}$$

where $\nabla u = (\frac{\partial u}{\partial x_1}, ..., \frac{\partial u}{\partial x_n})$ is the gradient of u (see [23] or [40]). The constant $\left|\frac{p}{n-p}\right|^p$ is sharp and is not attained in the corresponding Sobolev space, which is $W^{1,2}(\mathbb{R}^n)$ when $1 \le p < n$ and $W^{1,2}(\mathbb{R}^n \backslash \{0\})$ when $n < p < \infty$.

For $p = 2$ and $n > 2$, this inequality is also called the uncertainty principle. For $p = 2$ and $n = 2$, obviously, is trivial. However, in this case, if we weaken the singularity a bit by adding a logarithmic term or/and some extra conditions to the functions, for all $u \in C_0^\infty(\mathbb{R}^2 \backslash \{0\})$, we obtain the following inequalities (see [44]):

$$C \int_{\mathbb{R}^2} \frac{u^2(\mathbf{x})}{|\mathbf{x}|^2 \left(1 + \ln^2 |\mathbf{x}|\right)} d\mathbf{x} \leqslant \int_{\mathbb{R}^2} |\nabla u(\mathbf{x})|^2\, d\mathbf{x}, \text{ if } \int_{|\mathbf{x}|=1} u(\mathbf{x})\, d\mathbf{x} = 0 \tag{4}$$

and

$$C \int_{\mathbb{R}^2} \frac{u^2(\mathbf{x})}{|\mathbf{x}|^2} d\mathbf{x} \leqslant \int_{\mathbb{R}^2} |\nabla u(\mathbf{x})|^2\, d\mathbf{x}, \text{ if } \int_{|\mathbf{x}|=r} u(\mathbf{x})\, d\mathbf{x} = 0 \text{ for all } r > 0. \tag{5}$$

We note here that in the one-dimensional case, it was proved by Hardy in 1925 that for all p-integrable, $p > 1$ on $(0, 1)$, functions u, it holds

$$\int_0^1 \frac{|u(x)|^p}{d^{\,p}_{(0,1)}(x)} dx \leqslant \left(\frac{p}{p-1}\right)^p \int_0^1 |u'(x)|^p\, dx, \tag{6}$$

where $d_{(0,1)}(x) = \min(x, 1-x)$ (see in [21,22] and [9]).

In addition, Hardy showed that the constant is not attained, i.e. the variational problem has no minimizer. Furthermore, inequality (6) confirms that in the one-dimensional case no geometrical assumption is required on the domain.

It is quite natural to ask: Does an inequality of the form (6) continue to exist in the case of $\Omega \subset \mathbb{R}^n$ with $n \geqslant 2$? The answer to this question is positive, however, in regard to Hardy inequalities for domains in $\mathbb{R}^n$, $n \geqslant 2$ the situation is far more complicated than in the one-dimensional case and in general the best constant in (6) depends on the domain.

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