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Omega chaos and the specification property

Reeve Hunter [a,*], Brian E. Raines [b]

[a] *Department of Mathematics, University of Georgia, Athens, GA 30602-5016, USA*
[b] *Department of Mathematics, Baylor University, Waco, TX 76798-7328, USA*

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ABSTRACT

In this short paper we consider the connections between the specification property introduced by Bowen and ω-chaos introduced by Li. We show that if $f : X \to X$ is a continuous surjection of a compact metric space with the specification property and uniform expansion near a fixed point then the system (X, f) is ω-chaotic.

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1. Introduction and preliminaries

In 1971, R. Bowen introduced the *specification property* for a map on a compact metric space, [2]. We say that $f : X \to X$ has the *specification property* provided for every $\delta > 0$ there is some $N_\delta \in \mathbb{N}$ such that for $n \geq 2$ and for any n points $x_1, \ldots, x_n \in X$ and any sequence of natural numbers $a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_n \leq b_n$ with $a_i - b_{i-1} \geq N_\delta$ for $2 \leq i \leq n$ there is a periodic point $x \in X$ such that $d(f^j(x), f^j(x_i)) < \delta$ for $a_i \leq j \leq b_i$ and $1 \leq i \leq n$. The map has the *weak specification property* if the above holds with $n = 2$.

This is a strong property for a dynamical system. If the map is in addition expansive, Bowen showed that there is a unique equilibrium measure for any smooth potential, [3]. For an explanation of this result along with some other early results involving the specification property, see [6]. Despite the apparent strength of this property, Hofbauer showed that all continuous piecewise monotone maps of the interval have the weak specification property, [7], and Blokh showed that maps of the interval have the specification property if, and only if, they are topologically mixing, [1]. More recently, Buzzi has characterized the specification property in the context of piecewise continuous, piecewise monotone maps of the interval, [5].

In 2009, Lampart and Oprocha considered the weak specification property on symbolic dynamical systems, [8]. They characterized this property and the specification property on shift spaces. They also show

* Corresponding author.
E-mail addresses: Reeve.Hunter@uga.edu (R. Hunter), brian_raines@baylor.edu (B.E. Raines).

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that a shift space satisfying the weak specification property has a form of chaos called ω-chaos, introduced by S.H. Li in [9]. Let $f : X \to X$ be a map, and let $z \in X$. The ω-limit set of z is the set of all limit points of the orbit of z under f, and it is denoted by $\omega_f(z)$ or simply $\omega(z)$. We say that f has *ω-chaos* if there is an uncountable collection Ω of points of X such that for any $x, y \in \Omega$ with $x \neq y$ we have

(i) $\omega(x) \setminus \omega(y)$ is uncountable,
(ii) $\omega(x) \cap \omega(y) \neq \emptyset$,
(iii) $\omega(x)$ is not made up of only periodic points.

In this paper we partially answer a question raised by Lampart and Oprocha, [8], by showing that continuous surjections on compact metric spaces with the specification property and uniform expansion near a fixed point have ω-chaos. Our main theorem, specifically, is that if $f : X \to X$ is a continuous surjection with the specification property, X is compact metric, and there is some fixed point $s \in X$, $\eta > 0$ and $\lambda > 1$ such that if $d(s, y) < \eta$ then $d(s, f(y)) \geq \lambda d(s, y)$, then the system (X, f) has ω-chaos. We show this by encoding subsystems of (X, f) with subshifts from $(2^\omega, \sigma)$, the full shift on two symbols.

We now provide a few of the basic definitions and results needed throughout the paper. The interested reader should see [4] or [10] for additional background. In the following section we carefully construct some subsystems of (X, f) via an encoding of subshifts of $(2^\omega, \sigma)$ and prove the main theorem.

We use ω to stand for the set of natural numbers with 0. Let $A = \{0, 1\}$ have the discrete topology, and let $2^\omega = \prod_{n\in\omega} A$ have the product topology. We call 2^ω the *full shift on 2 symbols*. Let the *shift map* $\sigma : 2^\omega \to 2^\omega$ be defined by $\sigma(x_0, x_1 \dots) = (x_1, x_2 \dots)$. If $K \subseteq 2^\omega$ is closed and σ-invariant, i.e. $\sigma(K) = K$, then we call K a *subshift* of 2^ω. Given $\alpha \in 2^\omega$ and $k \in \omega$ we let

$$\alpha_{[0,k+1)} = \alpha_0, \alpha_1, \dots, \alpha_k.$$

Let $f : X \to X$ be a continuous map on a compact metric space (X, d). Let $K \subseteq X$. We call K *f-invariant* if $f(K) = K$. We say that K is *minimal* provided K is closed and f-invariant and there is no closed proper subset of K that is also f-invariant. Let $x \in X$. We say that x is *uniformly recurrent* provided for every $\varepsilon > 0$ there is some M_ε such that if $f^j(x) \in B_\varepsilon(x)$, for $j \geq 0$, then there is some $1 \leq k \leq M_\varepsilon$ such that $f^{j+k}(x) \in B_\varepsilon(x)$. It is a well known fact that K is minimal if, and only if, every point x of K is uniformly recurrent, [4]; moreover if x is uniformly recurrent then $\omega(x)$ is minimal.

It is not hard to construct an uncountable collection of points $\Lambda \subseteq 2^\omega$ that are uniformly recurrent with the property that if $x, y \in \Lambda$ and $x \neq y$ then $\omega(x) \cap \omega(y) = \emptyset$ and each $\omega(x)$ is uncountable. Thus there are uncountably many disjoint minimal subsets of 2^ω. We will use these points in what follows to find an uncountable collection of points witnessing ω-chaos.

2. Constructions and proof of the main theorem

In order to show f is ω-chaotic, we will exhibit an uncountable omega scrambled set. The construction of this set will rely on the fact mentioned above that 2^ω has an uncountable set of points whose ω-limit sets form an uncountable collection of minimal sets. We will prove our main theorem via a careful encoding of points from 2^ω into our dynamical system $f : X \to X$ with the specification property and uniform expansion near a fixed point.

We begin by fixing periodic points $t_0, t_1 \in X$. Then, for each $\alpha \in 2^\omega$, we will construct a point $x_\alpha \in X$ whose iterates under f get close to t_0 or t_1 in a pattern determined by the coordinates of α. The point x_α will also, under iterations of the map f, get close to the fixed point s mentioned in the theorem.