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# Simultaneous chaotic extensions for general operators on a Hilbert subspace

Kit C. Chan [a], Leonardo Pinheiro [b,*]

[a] Department of Mathematics and Statistics, Bowling Green State University, Bowling Green, OH 43403, United States
[b] Mathematics and Computer Science Department, Rhode Island College, Providence, RI 02908, United States

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ABSTRACT

For an infinite-codimensional closed subspace $M$ of a separable Hilbert space $H$, we show that every bounded linear operator $A : M \to H$ has a chaotic extension $T : H \to H$. As a generalization of this result, we further show that for any uniformly bounded sequence of linear operators $A_n : M \to H$, there exists a bounded linear operator $V : M^\perp \to H$ on the orthogonal complement $M^\perp$ of $M$ that simultaneously extends all operators $A_n$ to chaotic operators $A_n + V : H \to H$.



## 1. Introduction

On a separable, infinite-dimensional Hilbert space $H$, a bounded linear operator $T : H \to H$ is said to be *hypercyclic* if there is a vector $h$ in $H$ whose orbit $\text{orb}\,(T, h) = \{h, Th, T^2h, T^3h, \dots\}$ is dense in $H$. Such a vector $h$ is said to be a *hypercyclic vector* for $T$. Since every vector $T^nh$ in $\text{orb}\,(T, h)$ is necessarily a hypercyclic vector, every hypercyclic operator must have a dense set of hypercyclic vectors. In contrast, the orbit of a nonhypercyclic vector may be far from dense, and it may even be finite. For that situation, we say that a vector $g$ in $H$ is a *periodic point* for $T$ if there is a positive integer $n$ so that $T^ng = g$. When an operator $T : H \to H$ is hypercyclic and has a dense set of periodic points, we say that $T$ is a *chaotic operator* [8].

To show that an operator is hypercyclic, we have a well-known sufficient condition called the Hypercyclicity Criterion, which was first found by Kitai [14], and later rediscovered in much greater generality by Gethner and Shapiro [9]. A basic form of their Hypercyclicity Criterion for a separable, infinite-dimensional Hilbert space $H$ states the following:

* Corresponding author.
*E-mail addresses:* kchan@bgsu.edu (K.C. Chan), lpinheiro@ric.edu (L. Pinheiro).

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**Theorem 1.** *A bounded linear operator $T : H \to H$ is hypercyclic, if there is a dense subset $D$ of $H$ and a bounded linear operator $S : H \to H$ so that $TS =$ identity, and for every vector $x$ in $D$ we have $T^n x \to 0$ and $S^n x \to 0$.*

Indeed the hypotheses in Theorem 1 allow us to have a stronger conclusion that $T$ be *mixing*, which by definition means that for any nonempty open subsets $U$ and $V$ of $H$, there exists a positive integer $N$ such that $T^n(U) \cap V \neq \emptyset$ whenever $n \geq N$. In the present paper we are interested in the restriction $T|_M$ of a right invertible, mixing, chaotic operator $T$ onto a closed subspace $M$ of $H$ with $\dim(H/M) = \infty$. What properties does $T|_M$ possess? For that purpose, we say that a bounded linear operator $T : H \to H$ is an *extension* of a bounded linear operator $A : M \to H$, or simply $T$ *extends* $A$, when $T|_M = A$.

The first chaotic extension result for any arbitrary bounded linear operator $A : M \to M$ from an infinite-codimensional closed subspace $M$ back to $M$ itself was obtained by Grivaux [10, Proposition 1]. Her extension $T : H \to H$ was shown to satisfy the hypotheses of Theorem 1 above by Chan and Turcu [7, Theorem 2], and in particular $T$ is right invertible. This extension $T$ has the property that it takes some closed subspace $N$ of the orthogonal complement $M^\perp$ onto the codomain $M$ of $A$, and so $T$ is never injective. This naturally raised the question when $A$ can have an invertible chaotic extension $T$. In answering the question, Chan and Kadel [4, Theorem 2] found a chaotic extension $T$ with an additional property that $\ker T = \ker A$, in the case that $A$ has a closed range in $M$. Using this result, they [4, Corollary 4] showed that $A : M \to M$ has an invertible chaotic extension $T : H \to H$ if and only if $A$ is left invertible. Furthermore they [4, Corollary 5] showed that $A$ has a chaotic Fredholm extension $T$ if and only if $A$ is left semi-Fredholm.

Under certain conditions, $A$ may have a hypercyclic extension with an additional property. A bounded linear operator $T : H \to H$ is said to be *dual hypercyclic* if both $T$ and its adjoint $T^*$ are hypercyclic. The first example of a dual hypercyclic operator was offered by Salas [17], in answering the question of Herrero [13] whether such an operator can exist.

In terms of extension, Chan and Kadel [5, Theorem 3.2] showed that a bounded linear operator $A : M \to M$ with $\dim(H/M) = \infty$ has a dual hypercyclic extension $T : H \to H$ if and only if its adjoint $A^* : M \to M$ is hypercyclic. While this result shows that we cannot expect to have a dual hypercyclic extension for an arbitrary operator $A : M \to M$, Chan [3] showed that such an operator $A$ is the compression of a dual hypercyclic operator $T$. In other words, there exists a dual hypercyclic operator $T : H \to H$ so that if $P : H \to H$ is the orthogonal projection onto $M$ then $PTP = A$.

In Section 2 below, we construct a chaotic extension $T : H \to H$ for an arbitrarily prescribed bounded linear operator $A : M \to H$ that takes an infinite-codimensional closed subspace $M$ into the whole Hilbert space $H$. Furthermore, the chaotic extension $T$ satisfies the hypotheses of Theorem 1 and in particular $T$ is right invertible and mixing. Indeed the extension $T$ is also frequently hypercyclic, which by definition means that there exists a vector $h \in H$ such that for every nonempty open subset $U$ of $H$, the set $E = \{n \in \mathbb{N} : T^n x \in U\}$ has a positive lower density given by

$$\underline{\text{dens}}\,(E) = \liminf_{N\to\infty} \frac{\text{card}(E \cap [1, N])}{N},$$

where card denotes the cardinality.

The chaotic extension that we obtain for $A : M \to H$ generalizes the result of Grivaux [10, Proposition 1], and Chan and Turcu [7, Theorem 2] who offered the special case when $A$ takes $M$ back to $M$. As one can see in our construction of $T$ in Section 2, the part of the range of $A$ inside the orthogonal complement $M^\perp$ causes a lot of difficulties in our argument. To make sure the powers $T^k$ are working in the right way on $M^\perp$, we need to arrange orthonormal vectors in the range of $A$ in a very careful manner. For that we need a few lemmas before we provide a proof of the main theorem. Furthermore, the proof requires a norm estimation for $T^k f$ of a certain vector $f$. That argument is quite long and postponed to Section 6 as an appendix.