J. Math. Anal. Appl. • • • (• • • •) • • •–• • •

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Group classification of linear evolution equations

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ARTICLE INFO

Article history:
Received 30 May 2016
Available online xxxx
Submitted by W. Sarlet

Keywords:
Group classification of differential equations
Lie symmetries
Equivalence transformations
Lie reduction
Linear evolution equations
Exact solutions

ABSTRACT

The group classification problem for the class of (1+1)-dimensional linear rth order evolution equations is solved for arbitrary values of $r > 2$. It is shown that a related maximally gauged class of homogeneous linear evolution equations is uniformly semi-normalized with respect to linear superposition of solutions and hence the complete group classification can be obtained using the algebraic method. We also compute exact solutions for equations from the class under consideration using Lie reduction and its specific generalizations for linear equations.

1. Introduction

The investigation of higher-order evolution equations has been the subject of a considerable body of literature in recent years. Such equations naturally occur in the study of real-world problems, including water waves and solitary waves [18,21,22], thin film models [3,40], image processing [46] as well as integrable models [30].

While higher-order evolution equations typically arising in applications are nonlinear, there is substantial interest in studying higher-order linear evolution equations as well. In particular, the linearization of nonlinear evolution equations, which plays a key role in perturbation theory and stability analysis of these equations, leads to linear evolution equations of the same order with quite general variable coefficients. The study of a class of such linear equations within the framework of group analysis of differential equations is still a nontrivial problem since, in general, equation coefficients, which are interpreted as arbitrary elements of the class, are functions of several variables. Even the main problems of group analysis of differential equa-

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http://dx.doi.org/10.1016/j.jmaa.2016.11.020

tions – on Lie symmetries and on equivalence of equations – have been properly solved for (1+1)-dimensional evolution equations only in the case of order two [24,28,32].

Although symmetry methods play a more important role in the study of nonlinear differential equations than of linear ones, there are many papers devoted to various aspects of symmetry analysis of general linear systems of differential equations and their specific classes. This includes, in particular, general constraints imposed by the linearity of a system of differential equations on its Lie and point symmetries [5,16,17]; structure of algebras of generalized symmetries [42]; a specific advanced method for generating new solutions from known ones using Lie symmetries [8,11]; the description of conservation laws and potential symmetries of (1+1)-dimensional second-order linear evolution equations [38] as well as reduction operators and nonclassical reductions of these equations [12,35]; structure of Lie invariance algebras of linear systems of ordinary differential equations, group classification of such systems and admissible transformations between them [7,9,15,26,27,41].

In this paper we solve the group classification problem for the class of (1+1)-dimensional (in general, inhomogeneous) rth order ($r > 2$) linear evolution equations of the form

$$u_t = A^k(t,x)u_k + B(t,x), \quad A^r \neq 0. \tag{1}$$

Here and in the following, r is assumed to be an arbitrary but fixed integer greater than 2, and the summation over the repeated index k from 0 to r is implied. $u_k = \partial_k u = \partial^k u/\partial x^k$, where, by definition, $u_0 = u$. The functions $A^k = A^k(t,x)$ and $B = B(t,x)$ are smooth (e.g., analytic) functions of their arguments. The underlying field is real or complex. The consideration is within the local framework.

The group classification of equations of the form (1) for low values of r was the subject of several investigations. The case $r = 1$ is trivial since then the class (1) is the orbit of the degenerate equation $u_t = 0$ with respect to the corresponding equivalence group. The case $r = 2$ is specific among nontrivial values of r. It was exhaustively studied in the course of group classification of general second-order linear partial differential equations with two independent variables [24,32]. See also the review in [38, Section 2]. As specific and well studied, this case is excluded from the further consideration. Lie symmetries of third-order linear evolution equations were classified in [19]. Most recently, fourth-order equations of the form (1) were considered in [20]. The aim of this paper is to completely study Lie symmetries of linear evolution equations of arbitrary fixed order $r > 2$ using more advanced methods of group classification. This enhances the above results for orders three and four and extends them to an arbitrary (nontrivial) order.

Moreover, the class (1) is a subclass of the class of general (1+1)-dimensional evolution equations of order r. In [25], Magadeev studied contact symmetries of such equations with $r > 1$ up to contact equivalence. It was proved that if an evolution equation is not linearizable by a contact transformation then its contact symmetry algebra is of dimension not greater than $r + 5$. Magadeev classified, up to contact equivalence, algebras of vector fields in the space of t, x, u, u_x, which can serve as contact symmetry algebras for some evolution equations. At the same time, he did not present the form of equations, which admit these algebras. It is obvious that equations that are linearizable by contact transformations admit infinite-dimensional contact symmetry algebras. It is known that contact transformations between fixed linear evolution equations are prolongations of point transformations [45]. This is why the classification of contact symmetries of such equations up to contact equivalence degenerates to the classification of point symmetries of such equations up to point equivalence. As a result, the group classification of the class (1) completes Magadeev's studies of contact symmetries of evolution equations.

The further consideration is the following. In Section 2 we present a brief review of the algebraic method of group classification. Special attention is paid to the version of this method for uniformly semi-normalized classes of differential equations, which is relevant for this paper. Section 3 is devoted to the computation of the equivalence groupoids and the equivalence groups of the class (1) and some of its gauged subclasses. We show that the class (1) and two of its subclasses singled out by gauging subleading coefficients, $A^r = 1$ and

Please cite this article in press as: A. Bihlo, R.O. Popovych, Group classification of linear evolution equations, J. Math. Anal. Appl. (2017), http://dx.doi.org/10.1016/j.jmaa.2016.11.020